Problem 11400
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Proposed by P. Bracken (USA).

Let ζ be the Riemann zeta function. Evaluate

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}$$

in closed form.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $0 \leq x < 1$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^n = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{\sqrt{x}}{k}\right)^{2n} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\sqrt{x}}{k}\right)^{2n} = -\log \prod_{k=1}^{\infty} \left(1 - \left(\frac{\sqrt{x}}{k}\right)^2\right) = -\log\left(\frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}}\right).$$

Hence, because of uniform convergence of the series, we get the desired closed form

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)} &= \int_0^1 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^n = -\int_0^1 \log\left(\frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}}\right) dx \\
&= -\int_0^1 \log\left(\sin(\pi\sqrt{x})\right) dx + \int_0^1 \log(\pi)\ dx + \frac{1}{2}\int_0^1 \log(x)\ dx \\
&= -\frac{2}{\pi^2}\int_0^{\pi} \log(\sin(t))\, t\, dt + \log(\pi) - \frac{1}{2} \\
&= \log(2) + \log(\pi) - \frac{1}{2} = \log(2\pi) - \frac{1}{2} \approx 1.337877.
\end{aligned}$$

Note that the nontrivial integral can be computed by exploiting symmetry:

$$I = \int_0^{\pi} \log(\sin(t))\, t\, dt = \int_0^{\pi} \log(\sin(\pi - t))\, (\pi - t)\, dt = \pi \int_0^{\pi} \log(\sin(t))\ dt - I$$

and

$$\begin{aligned}
J &= \int_0^{\pi} \log(\sin(t))\ dt = \int_0^{\pi} \log(2\sin(t/2)\cos(t/2))\ dt \\
&= \pi\log(2) + 2\int_0^{\pi/2} \log\sin(t)\, dt + 2\int_0^{\pi/2} \log\cos(t)\, dt \\
&= \pi\log(2) + 2\int_0^{\pi/2} \log\sin(t)\, dt + 2\int_0^{\pi/2} \log\sin(\pi/2 - t)\, dt = \pi\log(2) + 2J
\end{aligned}$$

therefore $J = -\pi\log(2)$ and $I = \pi J/2 = -\pi^2\log(2)/2$. $\square$