Problem 11394
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Proposed by K.S. Bhanu (India).

A fair coin is tossed n times, with $n \geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that the expected values of R, X and $R \cdot X$ on $\{H,T\}^n$ are

$$E_n(R) = (n+1)/2, \quad E_n(X) = (n+2)/8, \quad E_n(R \cdot X) = (n^2+5n+2)/16$$

and therefore

$$\mathrm{Cov}_n(R,X) = E_n(R \cdot X) - E_n(R) \cdot E_n(X) = (n^2+5n+2)/16 - (n+2)(n+1)/16 = n/8.$$

First we give some notations: let $t \in \{H,T\}^m$ and let Y be a random variable on $\{H,T\}^{n+m}$, then we define

$$y(n,t) = \sum_{s \in \{H,T\}^n} Y(st)$$

where st is the sequence of $n+m$ tossings whose first part is s and the second part is t. When t is empty we agree that $y(n,t)$ can be written as $y(n)$. Note that $E_n(Y) = y(n)/2^n$.

(a) Since R counts the number of runs of the same face, we have that

$$\begin{aligned}
R(sTT) &= R(sT) & \forall s \in \{H,T\}^n &\implies r(n,TT) = r(n,T)\\
R(sTH) &= R(sT)+1 & \forall s \in \{H,T\}^n &\implies r(n,TH) = r(n,T)+2^n\\
R(sHT) &= R(sH)+1 & \forall s \in \{H,T\}^n &\implies r(n,HT) = r(n,H)+2^n\\
R(sHH) &= R(sH) & \forall s \in \{H,T\}^n &\implies r(n,HH) = r(n,H)
\end{aligned}$$

therefore

$$\begin{aligned}
r(n+1,T) &= r(n,HT)+r(n,TT) = r(n,H)+2^n+r(n,T) = r(n+1)+2^n\\
r(n+1,H) &= r(n,HH)+r(n,TH) = r(n,H)+r(n,T)+2^n = r(n+1)+2^n.
\end{aligned}$$

Finally, since

$$r(n+2) = r(n+1,T)+r(n+1,H) = 2r(n)+2^{n+1}$$

and

$$E_2(R) = r(2)/4 = (R(TT)+R(TH)+R(HT)+R(HH))/4 = (1+2+2+1)/4 = 3/2$$

then

$$E_n(R) = E_{n-1}(R)+1/2 = (n+1)/2.$$

(b) Since X counts the number of isolated H's, we have that

$$\begin{aligned}
X(sT) &= X(s) & \forall s \in \{H,T\}^n &\implies x(n,T) = x(n)\\
X(sTH) &= X(sT)+1 & \forall s \in \{H,T\}^n &\implies x(n,TH) = x(n,T)+2^n\\
X(sTHH) &= X(sTH)-1 & \forall s \in \{H,T\}^n &\implies x(n,THH) = x(n,TH)-2^n\\
X(sHHH) &= X(sHH) & \forall s \in \{H,T\}^n &\implies x(n,HHH) = x(n,HH)
\end{aligned}$$

therefore

$$\begin{aligned} x(n+2,H) &= x(n+1,TH) + x(n,THH) + x(n,HHH) \\ &= x(n+1,T) + 2^{n+1} + x(n,TH) - 2^n + x(n,HH) = x(n+2) + 2^n. \end{aligned}$$

Finally, since

$$x(n+3) = x(n+2,T) + x(n+2,H) = 2x(n+2) + 2^n$$

and

$$E_2(X) = (X(TT) + X(TH) + X(HT) + X(HH))/4 = (0+1+1+0)/4 = 1/2$$

then

$$E_n(X) = E_{n-1}(X) + 1/8 = (n+2)/8.$$

(c) Let $Z = R \cdot X$ then

$$\begin{aligned} Z(sTT) &= Z(sT) & \forall s \in \{H,T\}^n &\implies z(n,TT) = z(n,T) \\ Z(sTH) &= Z(sT) + R(sT) + X(sT) + 1 & \forall s \in \{H,T\}^n &\implies z(n,TH) = z(n,T) + r(n,T) + x(n,T) + 2^n \\ Z(sHT) &= Z(sH) + X(sH) & \forall s \in \{H,T\}^n &\implies z(n,HT) = z(n,H) + x(n,H) \\ Z(sTHH) &= Z(sTH) - R(sTH) & \forall s \in \{H,T\}^n &\implies z(n,THH) = z(n,TH) - r(n,TH) \\ Z(sHHH) &= Z(sHH) & \forall s \in \{H,T\}^n &\implies z(n,HHH) = z(n,HH) \end{aligned}$$

therefore

$$\begin{aligned} z(n+1,T) &= z(n,TT) + z(n,HT) = z(n,T) + z(n,H) + x(n,H) = z(n+1) + x(n,H) \\ z(n+1,HH) &= z(n,THH) + z(n,HHH) = z(n,TH) - r(n,TH) + z(n,HH) = z(n+1,H) - r(n,TH). \end{aligned}$$

Since $r(n+1,T) = r(n+1,H)$ and $r(n,HH) = r(n,H)$ then

$$\begin{aligned} z(n+2,H) &= z(n+1,TH) + z(n+1,HH) \\ &= z(n+1,T) + r(n+1,T) + x(n+1,T) + 2^{n+1} + z(n+1,H) - r(n,TH) \\ &= z(n+2) + r(n+1,T) + x(n+1,T) + 2^{n+1} - r(n+1,H) + r(n,HH) \\ &= z(n+2) + x(n+1,T) + 2^{n+1} + r(n,H). \end{aligned}$$

Finally, since

$$\begin{aligned} z(n+3) &= z(n+2,T) + z(n+2,H) \\ &= z(n+2) + x(n+1,H) + z(n+2) + x(n+1,T) + 2^{n+1} + r(n,H) \\ &= 2z(n+2) + x(n+2) + 2^{n+1} + r(n) + 2^{n-1} \\ &= 2z(n+2) + 2^{n-1}(n+4) + 2^{n-1}(n+1) + 5\,2^{n-1} \\ &= 2z(n+2) + 2^n n + 5\,2^n \end{aligned}$$

and

$$E_2(Z) = z(2)/4 = (Z(TT) + Z(TH) + Z(HT) + Z(HH))/4 = (0+2+2+0)/4 = 1$$

then

$$E_n(Z) = E_{n1}(Z) + n/8 + 1/4 = (n^2 + 5n + 2)/16.$$

$\square$