Problem 11384
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Proposed by M. Omarjee (France).

Let p_n denote the nth prime. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n}$$

converges.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since $p_n = n\log n + O(n\log(\log n))$, then for M, N sufficiently large there is a positive constant C such that

$$\left|\sum_{n=N}^{M} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{p_n} - \sum_{n=N}^{M} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{n\log n}\right| \leq \sum_{n=N}^{M} \left|\frac{1}{p_n} - \frac{1}{n\log n}\right| \leq C\sum_{n=N}^{M} \frac{\log(\log n)}{n(\log n)^2}$$

which tends to zero as M, N goes to infinity because the series

$$\sum_{n=2}^{\infty} \frac{\log(\log n)}{n(\log n)^2}$$

converges. Hence it suffices to prove that the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{n\log n}$$

converges. For any $M > N > 2$

$$\begin{aligned}
\sum_{n=N}^{M} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{n\log n} &= \sum_{n=N}^{M} \frac{A_n - A_{n-1}}{n\log n} = \sum_{n=N}^{M} \frac{A_n}{n\log n} - \sum_{n=N}^{M} \frac{A_{n-1}}{n\log n} \\
&= \sum_{n=N}^{M} \frac{A_n}{n\log n} - \sum_{n=N-1}^{M-1} \frac{A_n}{(n+1)\log(n+1)} \\
&= \sum_{n=N}^{M} A_n\left(\frac{1}{n\log n} - \frac{1}{(n+1)\log(n+1)}\right) + \frac{A_M}{M\log M} - \frac{A_{N-1}}{N\log N}.
\end{aligned}$$

Let $a = \lfloor\sqrt{n}\rfloor$ then

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} (-1)^{\lfloor\sqrt{k}\rfloor} = \sum_{j=1}^{a-1} \sum_{k=j^2}^{(j+1)^2-1} (-1)^j + \sum_{k=a^2}^{n} (-1)^a \\
&= \sum_{j=1}^{a-1} (-1)^j(2j+1) + \sum_{k=a^2}^{n} (-1)^a = -(1 + (-1)^a a) + (-1)^a(n - (a^2 - 1)).
\end{aligned}$$

Hence

$$|A_n| \leq \sqrt{n} + 2 = O(\sqrt{n}).$$

Moreover

$$\frac{1}{n\log n} - \frac{1}{(n+1)\log(n+1)} = O\left(\frac{1}{n^2\log n}\right).$$

Therefore for M, N sufficiently large there is a positive constant C such that

$$\left|\sum_{n=N}^{M} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{n\log n}\right| \leq C\left(\sum_{n=N}^{M} \frac{1}{n^{3/2}\log n} + \frac{1}{\sqrt{M}\log M} + \frac{1}{\sqrt{N}\log N}\right)$$

which tends to zero as M, N goes to infinity and therefore the series converges. $\square$