**Problem 11382**
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Proposed by R. Tauraso (Italy).

*For $k \geq 1$, let $H_k$ be the $k$th harmonic number, $H_k = \sum_{j=1}^{k} 1/j$. Show that if $p$ is prime and $p > 5$, then*

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \pmod{p^2}.$$

*(Two rationals are congruent mod $d$ if their difference can be expressed as a reduced fraction of the form $da/b$, with $b$ relatively prime to $a$ and $d$.)*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$H_{k-1}^3 = \left(H_k - \frac{1}{k}\right)^3 = H_k^3 - 3\frac{H_k^2}{k} + 3\frac{H_k}{k^2} - \frac{1}{k^3}$$

therefore

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} = \frac{1}{3}\left(\sum_{k=1}^{p-1} \left(H_k^3 - H_{k-1}^3\right) - \sum_{k=1}^{p-1} \frac{1}{k^3}\right) = \frac{1}{3}\left(H_{p-1}^3 - H_{p-1,3}\right)$$

Since

$$H_{p-1,d} = \sum_{k=1}^{p-1} \frac{1}{k^d} = 0 \mod p^2$$

for any prime number $p$ and for all odd positive numbers $d$ such that $d < p-2$ (see for example p.104 *An Introduction to the Theory of Numbers* by Hardy and Wright) then

$$H_{p-1,1} = H_{p-1} = 0 \mod p^2 \quad \text{and} \quad H_{p-1,3} = 0 \mod p^2$$

if and only if $3 < p-2$ that is $p > 5$. Therefore for $p > 5$

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} = \frac{1}{3}\left(H_{p-1}^3 - H_{p-1,3}\right) = 0 \mod p^2.$$

□