Problem 11373
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Proposed by E. Deutsch (USA).

Let S_n be the symmetric group on $\{1,\dots,n\}$. By the canonical cycle decomposition of an element π of S, we mean the cycle decomposition of π in which the largest entry of each cycle is at the beginning of that cycle, and the cycles are arranged in increasing order of their first elements. Let $\psi_n : S_n \to S_n$ be the mapping that associates to each $\pi \in S_n$ the permutation whose one-line representation is obtained by removing the parentheses from the canonical cycle decomposition of π. Describe the fixed points of ψ_n and find their number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\pi \in S_n$. According to the canonical cycle decomposition, the cycle of n begins with n and ends with $\pi^{-1}(n)$. Therefore, if π is a fixed point of ψ_n then

$$\pi(n) = \pi^{-1}(n) \in \{n-1, n\}.$$

Hence the the cycle of n is (n) or $(n, n-1)$. In the former case, π restricted to S_{n-1} is a permutation of $\{1,\dots,n-1\}$ and it is a fixed point of ψ_{n-1}, whereas in the latter case, π restricted to S_{n-2} is a permutation of $\{1,\dots,n-2\}$ and it is a fixed point of ψ_{n-2}. This implies that the following recurrence holds

$$|\mathrm{Fix}(\psi(n))| = |\mathrm{Fix}(\psi(n-1))| + |\mathrm{Fix}(\psi(n-2))|.$$

Since

$$|\mathrm{Fix}(\psi(1))| = |\{(1)\}| = 1 \quad \text{and} \quad |\mathrm{Fix}(\psi(2))| = |\{(1)(2), (2,1)\}| = 2$$

then

$$|\mathrm{Fix}(\psi(n))| = (n+1)\text{-th Fibonacci number}.$$

Finally we can say that a permutation $\pi \in S_n$ is a fixed point of $\psi(n)$ if and only if it has all cycles of size at most 2 and each cycle of size 2 is made of two consecutive numbers. This description gives also a bijection of $\mathrm{Fix}(\psi(n))$ with the domino tilings of a strip $2 \times n$: a vertical domino in position k corresponds to the cycle (k), two horizontal dominoes in position k and $k+1$ correspond to the cycle $(k+1, k)$. □