Problem 11369
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Proposed by D. Knuth (USA).

Prove that for all real t, and all $\alpha \geq 2$,

$$e^{\alpha t} + e^{-\alpha t} - 2 \leq (e^t + e^{-t})^{\alpha} - 2^{\alpha}.$$

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Equality holds trivially for $t = 0$ or for $\alpha = 2$.
We will show that for $t \neq 0$ and $\alpha > 2$ then

$$e^{\alpha t} + e^{-\alpha t} - 2 < (e^t + e^{-t})^{\alpha} - 2^{\alpha}.$$

Let

$$f(x) = (x + x^{-1})^{\alpha} - x^{\alpha} - x^{-\alpha}$$

then, by letting $x = e^t$, it suffices to show that $f(x) > f(1)$ for $x > 0$ and $x \neq 1$.
Since $f(x) = f(x^{-1})$ it suffices to prove that f is strictly decreasing in $(0, 1)$.
Assume that $0 < x < 1$ then

$$f'(x) = \alpha(x + x^{-1})^{\alpha-1}(1 - x^{-2}) - \alpha x^{\alpha-1} + \alpha x^{-\alpha+1} < 0$$

is equivalent to

$$(1 + x^2)^{\alpha-1} > \frac{1 - x^{2\alpha}}{1 - x^2}.$$

By Bernoulli inequality ($(1 + t)^a > 1 + at$ for $a > 1$ and for any $t \geq -1$ and $t \neq 0$) then

$$(1 + x^2)^{\alpha-1} > 1 + (\alpha - 1)x^2,$$

and

$$1 + (\alpha - 1)x^2 > \frac{1 - x^{2\alpha}}{1 - x^2} = 1 + \left(\frac{1 - x^{2(\alpha-1)}}{1 - x^2}\right) x^2$$

holds as soon as

$$\alpha - 1 > \frac{1 - x^{2(\alpha-1)}}{1 - x^2}$$

that is

$$x^{2(\alpha-1)} > 1 + (\alpha - 1)(x^2 - 1)$$

which holds again by Bernoulli inequality

$$x^{2(\alpha-1)} = (1 + (x^2 - 1))^{\alpha-1} > 1 + (\alpha - 1)(x^2 - 1).$$

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