Problem 11364
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Proposed by P. P. Dalyay (Hungary).

Let p be a prime greater than 3, and t the integer nearest $p/6$.
(a) Show that if $p = 6t + 1$, then

$$(p-1)!\left(\sum_{j=0}^{2t-1}\frac{(-1)^j}{3j+1}+\sum_{j=0}^{2t-1}\frac{(-1)^j}{3j+2}\right)\equiv 0 \pmod{p}.$$

(b) Show that if $p = 6t - 1$, then

$$(p-1)!\left(\sum_{j=0}^{2t-1}\frac{(-1)^j}{3j+1}+\sum_{j=0}^{2t-2}\frac{(-1)^j}{3j+2}\right)\equiv 0 \pmod{p}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\omega = e^{2\pi i/3}$, then

$$\sum_{k=1}^{p-1}\omega^k\binom{p}{k} = (1+\omega)^p - 1 - \omega^p = e^{(6t\pm1)\pi i/3} - 1 - e^{2(6t\pm1)\pi i/3} = e^{\pm\pi i/3} - 1 - e^{\pm 2\pi i/3} = 0$$

On the other hand, if $p = 6t + 1$ then the rhs can be written as follows

$$\begin{aligned}
\sum_{k=1}^{p-1}\omega^k\binom{p}{k} &= \sum_{j=1}^{2t}\omega^{3j}\binom{p}{3j}+\sum_{j=1}^{2t-1}\omega^{3j+1}\binom{p}{3j+1}+\sum_{j=0}^{2t-1}\omega^{3j+2}\binom{p}{3j+2}\\
&= \sum_{j=1}^{2t}\frac{p}{3j}\binom{p-1}{3j-1}+\sum_{j=0}^{2t-1}\omega\frac{p}{3j+1}\binom{p-1}{3j}+\sum_{j=0}^{2t-1}\omega^2\frac{p}{3j+2}\binom{p-1}{3j+1}\\
&= p\left(\sum_{j=1}^{2t}\frac{1}{3j}\binom{p-1}{3j-1}+\sum_{j=0}^{2t-1}\frac{\omega}{3j+1}\binom{p-1}{3j}+\sum_{j=0}^{2t-1}\frac{\overline{\omega}}{3j+2}\binom{p-1}{3j+1}\right)
\end{aligned}$$

Therefore, by taking the imaginary part we have that

$$\sum_{j=0}^{2t-1}\frac{1}{3j+1}\binom{p-1}{3j}-\sum_{j=0}^{2t-1}\frac{1}{3j+2}\binom{p-1}{3j+1}=0.$$

Since for $0 \le k \le p-1$

$$(p-1)!\binom{p-1}{k} = (p-1)(p-2)\cdots(k+1)\cdot(p-1)(p-2)\cdots(p-k) \equiv (-1)^{k+1} \pmod{p}$$

then the above identity gives (a).
If $p = 6t - 1$ then the rhs of the first equation can be written as follows

$$\begin{aligned}
\sum_{k=1}^{p-1}\omega^k\binom{p}{k} &= \sum_{j=1}^{2t-1}\omega^{3j}\binom{p}{3j}+\sum_{j=1}^{2t-1}\omega^{3j+1}\binom{p}{3j+1}+\sum_{j=0}^{2t-2}\omega^{3j+2}\binom{p}{3j+2}\\
&= \sum_{j=1}^{2t-1}\frac{p}{3j}\binom{p-1}{3j-1}+\sum_{j=0}^{2t-1}\omega\frac{p}{3j+1}\binom{p-1}{3j}+\sum_{j=0}^{2t-2}\omega^2\frac{p}{3j+2}\binom{p-1}{3j+1}\\
&= p\left(\sum_{j=1}^{2t}\frac{1}{3j}\binom{p-1}{3j-1}+\sum_{j=0}^{2t-1}\frac{\omega}{3j+1}\binom{p-1}{3j}+\sum_{j=0}^{2t-2}\frac{\overline{\omega}}{3j+2}\binom{p-1}{3j+1}\right)
\end{aligned}$$

Hence, by taking the imaginary part we have that

$$\sum_{j=0}^{2t-1} \frac{1}{3j+1}\binom{p-1}{3j} - \sum_{j=0}^{2t-2} \frac{1}{3j+2}\binom{p-1}{3j+1} = 0$$

which gives (b). $\square$