

Problem 11362

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A bit string arc diagram is an undirected graph in which the vertices are the positions in a single string of bits and the edges are called arcs due to the visual representation in which they are drawn joining positions in the string. To be a good diagram, arcs must occur only between unequal bits, and each bit may be the left endpoint of at most one arc. There are six good diagrams on two bits, four with no arc and two with a single arc.

How many good diagrams are there on n bits?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that on n bits there are $g_n = (n + 1)!$ good diagrams.

All good diagrams on n bits for $n > 2$ can be constructed in a unique way as follows:

- (1) take a good diagrams on $n - 1$ bits;
- (2) add a vertex to the left;
- (3) join this vertex to one of the bits on the right (there are $n - 1$ choices) and add a label 0 or 1 whether it has been connected with a 1 or a 0 or do not join this vertex to any other bit and add a label 0 or 1 (2 more choices).

The last step can be done in $(n - 1) + 2 = n + 1$ ways, therefore we have that

$$g_n = (n + 1) \cdot g_{n-1},$$

and since we already know that $g_2 = 3! = 6$, it follows that $g_n = (n + 1)!$. □