Problem 11360
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Let f and g be continuous real-valued functions on $[0,1]$ satisfying the condition $\int_0^1 fg = 0$. Show that

$$\int_0^1 f^2 \int_0^1 g^2 \geq 4\left(\int_0^1 f \int_0^1 g\right)^2$$

and

$$\int_0^1 f^2 \left(\int_0^1 g\right)^2 + \int_0^1 g^2 \left(\int_0^1 f\right)^2 \geq 4\left(\int_0^1 f \int_0^1 g\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$\int_0^1 f^2 = A, \int_0^1 g^2 = B, \int_0^1 f = a, \int_0^1 g = b,$$

we will prove that

$$AB \geq Ab^2 + Ba^2 \geq 4a^2b^2.$$

By Cauchy-Schwarz inequality, $B \geq b^2$ and equality holds iff g is constant. In that case, $\int_0^1 fg = 0$ implies that $a = 0$ and the inequalities hold trivially. So we can assume that $B > b^2$.
By Cauchy-Schwarz inequality, for any real t we have that

$$\int_0^1 1 \cdot \int_0^1 (f + tg)^2 \geq \left(\int_0^1 [1 \cdot (f + tg)]\right)^2$$

that is, since $\int_0^1 fg = 0$,

$$A + Bt^2 \geq a^2 + 2abt + b^2t^2$$

or

$$A \geq \sup_{t \in \mathbb{R}} \{a^2 + 2abt - (B - b^2)t^2\}.$$

Since $B > b^2$, then the second degree polynomial assume its maximum value for $t = ab/(B - b^2)$ and we have that

$$A \geq a^2 + 2ab\frac{ab}{(B - b^2)} - (B - b^2)\frac{a^2b^2}{(B - b^2)^2} = a^2 + \frac{a^2b^2}{(B - b^2)}$$

that is

$$AB \geq Ab^2 + Ba^2.$$

Finally, again by Cauchy-Schwarz inequality,

$$AB \geq Ab^2 + Ba^2 = \int_0^1 (bf + ag)^2 \geq \left(\int_0^1 (bf + ag)\right)^2 = (2ab)^2 = 4a^2b^2.$$

□