Problem 11356
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Prove that for any positive integer n,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The equality can be easily written in this way:

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2(n-k)}{n-k} = \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}.$$

This identity is well known (see *Combinatorial Identities* by Riordan) and it can be proved via generating functions. Since

$$\frac{1}{\sqrt{1-4z^2}} = \sum_{n=0}^{\infty}\binom{2k}{k}z^{2k}$$

then, by integrating it, we have that

$$\frac{\arcsin(2z)}{2} = \sum_{n=0}^{\infty}\frac{1}{2k+1}\binom{2k}{k}z^{2k+1}$$

therefore

$$\frac{\arcsin(2z)}{2\sqrt{1-4z^2}} = \sum_{n=0}^{\infty} a_n z^{2n+1} \quad \text{where } a_n = \sum_{k=0}^{n}\frac{1}{2k+1}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

Let

$$f(z) = \sum_{n=0}^{\infty} b_n z^{2n+1} \quad \text{where } b_n = \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-1}$$

then since $b_0 = 1$ and $(2n+3)b_{n+1} = 8(n+1)b_n = 4(2n+1)b_n + 4b_n$ then f is the unique solution in a neighborhood of $z = 0$ of the Cauchy problem

$$f'(z) - 1 = 4z^2 f'(z) + 4zf(z) \quad \text{with } f(0) = 1.$$

Hence the equality holds as soon as we verify that $f(z) = (1/2)\arcsin(2z)/\sqrt{1-4z^2}$ is the solution of the above problem: $f(0) = 1$ and

$$(1-4z^2)f'(z) = (1-4z^2)\left(\frac{2}{2\sqrt{1-4z^2}}\cdot\frac{1}{\sqrt{1-4z^2}} + \arcsin(2z)\frac{4z}{2(1-4z^2)^{3/2}}\right) = 1 + 4zf(z).$$

□