Problem 11354
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Proposed by M. Beck and A. Berkovich (USA).

Find a polynomial f in two variables such that for all pairs (s,t) of relatively prime positive integers,

$$\sum_{m=1}^{s-1}\sum_{n=1}^{t-1}|mt-ns| = f(s,t).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that $f(s,t)=\dfrac{(s-1)(t-1)(2st-s-t-1)}{6}$:

$$\begin{aligned}
\sum_{m=1}^{s-1}\sum_{n=1}^{t-1}|mt-ns| &= \sum_{m'=1}^{s-1}\sum_{n=1}^{t-1}|(s-m')t-ns| \\
&= \sum_{m'=1}^{s-1}\sum_{n=1}^{t-1}|st-s-t-((m'-1)t+(n-1)s)| \\
&= \sum_{a=0}^{s-2}\sum_{b=0}^{t-2}|st-s-t-(at+bs)| \\
&= 2\sum_{d=0}^{st-s-t-1}(st-s-t-d)\,p_{\{s,t\}}(d) \\
&= 2(st-s-t)\sum_{d=0}^{st-s-t-1}p_{\{s,t\}}(d)-2\sum_{d=1}^{st-s-t-1}dp_{\{s,t\}}(d) \\
&= (st-s-t)(s-1)(t-1)-\frac{(s-1)(t-1)(4st-5s-5t+1)}{6} \\
&= \frac{(s-1)(t-1)(2st-s-t-1)}{6}
\end{aligned}$$

where

$$p_{\{s,t\}}(d)=|\{(a,b)\in\mathbb{N}^2 : at+bs=d\}|$$

and

$$\sum_{d=0}^{st-s-t-1}p_{\{s,t\}}(d)=\frac{(s-1)(t-1)}{2},\quad \sum_{d=1}^{st-s-t-1}dp_{\{s,t\}}(d)=\frac{(s-1)(t-1)(4st-5s-5t+1)}{12}.$$

Now we prove the last two formulas by using various results contained in *Computing the Continuous Discretely* by M. Beck and S. Robins. Let $\zeta_s=e^{2\pi i/s}$ and $\zeta_t=e^{2\pi i/t}$ then by (1.7)

$$p_{\{s,t\}}(d)=\frac{1}{2s}+\frac{1}{2t}+\frac{d}{st}+\frac{1}{s}\sum_{k=1}^{s-1}\frac{1}{(1-\zeta_s^{kt})\zeta_s^{kd}}+\frac{1}{t}\sum_{k=1}^{t-1}\frac{1}{(1-\zeta_t^{ks})\zeta_t^{kd}}$$

Since

$$\sum_{d=0}^{st-s-t-1}\frac{1}{\zeta_s^{kd}}=\frac{\zeta_s^{-(st-s-t)k}-1}{\zeta_s^{-k}-1}=-\frac{(1-\zeta_s^{kt})\zeta_s^{k}}{1-\zeta_s^{k}}.$$

Then by (1.10)

$$\begin{aligned}\sum_{d=0}^{st-s-t-1} p_{\{s,t\}}(d) &= (st-s-t)\left(\frac{1}{2s}+\frac{1}{2t}+\frac{st-s-t-1}{2st}\right)-\frac{1}{s}\sum_{k=1}^{s-1}\frac{\zeta_s^k}{(1-\zeta_s^k)}-\frac{1}{t}\sum_{k=1}^{t-1}\frac{\zeta_t^k}{(1-\zeta_t^k)}\\ &= \frac{st-s-t}{2}\left(1-\frac{1}{st}\right)+1-\frac{1}{2s}-\frac{1}{2t}=\frac{(s-1)(t-1)}{2}.\end{aligned}$$

Since

$$\sum_{d=0}^{st-s-t-1}\frac{d}{\zeta_s^{kd}}=\frac{(st-s-t)\zeta_s^k}{1-\zeta_s^k}-\frac{(st-s-t)(1-\zeta_s^{kt})\zeta_s^k}{1-\zeta_s^k}+\frac{\zeta_s^k(1-\zeta_s^{kt})}{(1-\zeta_s^k)^2}.$$

Then by (8.1)

$$\begin{aligned}\sum_{d=0}^{st-s-t-1}\frac{1}{s}\sum_{k=1}^{s-1}\frac{d}{(1-\zeta_s^{kt})\zeta_s^{kd}} &= \frac{(st-s-t)}{s}\sum_{k=1}^{s-1}\frac{\zeta_s^k}{(1-\zeta_s^{kt})(1-\zeta_s^k)}\\ &\quad-\frac{(st-s-t)}{s}\sum_{k=1}^{s-1}\frac{\zeta_s^k}{1-\zeta_s^k}+\frac{1}{s}\sum_{k=1}^{s-1}\frac{\zeta_s^k}{(1-\zeta_s^k)^2}\\ &= (st-s-t)\left(-S(t,s)-\frac{1}{4}+\frac{1}{4s}\right)\\ &\quad-(st-s-t)\left(-\frac{1}{2}+\frac{1}{2s}\right)-\frac{1}{12}\left(s-\frac{1}{s}\right)\\ &= (st-s-t)\left(-S(t,s)+\frac{1}{4}-\frac{1}{4s}\right)-\frac{1}{12}\left(s-\frac{1}{s}\right)\end{aligned}$$

where

$$S(t,s)=\frac{1}{4s}\sum_{k=1}^{s-1}\frac{1+\zeta_s^k}{1-\zeta_s^k}\cdot\frac{1+\zeta_s^{-kt}}{1-\zeta_s^{-kt}}$$

is a Dedekind sum. In the same way,

$$\sum_{d=0}^{st-s-t-1}\frac{1}{t}\sum_{k=1}^{t-1}\frac{d}{(1-\zeta_t^{ks})\zeta_t^{kd}} = (st-s-t)\left(-S(s,t)+\frac{1}{4}-\frac{1}{4t}\right)-\frac{1}{12}\left(t-\frac{1}{t}\right).$$

Therefore, by Dedekind's reciprocity law (8.5)

$$S(t,s)+S(s,t)=\frac{1}{12}\left(\frac{s}{t}+\frac{t}{s}+\frac{1}{st}\right)-\frac{1}{4}$$

and we have that

$$\begin{aligned}\sum_{d=0}^{st-s-t-1}\left(\frac{1}{s}\sum_{k=1}^{s-1}\frac{d}{(1-\zeta_s^{kt})\zeta_s^{kd}}+\frac{1}{t}\sum_{k=1}^{t-1}\frac{d}{(1-\zeta_t^{ks})\zeta_t^{kd}}\right) &= (st-s-t)\left(-\frac{1}{12}\left(\frac{s}{t}+\frac{t}{s}+\frac{1}{st}\right)+\frac{3}{4}-\frac{1}{4s}-\frac{1}{4t}\right)\\ &\quad-\frac{1}{12}\left(t+s-\frac{1}{t}-\frac{1}{s}\right).\end{aligned}$$

Finally, after some calculations, we find that

$$\begin{aligned}\sum_{d=1}^{st-s-t-1} dp_{\{s,t\}}(d) &= \left(\frac{1}{2s}+\frac{1}{2t}\right)\sum_{d=1}^{st-s-t-1} d+\frac{1}{st}\sum_{d=1}^{st-s-t-1} d^2\\ &\quad+\sum_{d=0}^{st-s-t-1}\left(\frac{1}{s}\sum_{k=1}^{s-1}\frac{d}{(1-\zeta_s^{kt})\zeta_s^{kd}}+\frac{1}{t}\sum_{k=1}^{t-1}\frac{d}{(1-\zeta_t^{ks})\zeta_t^{kd}}\right)\\ &= \frac{(s-1)(t-1)(4st-5s-5t+1)}{12}.\end{aligned}$$

□