Problem 11344
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Proposed by A. Stadler (Switzerland).

Let μ be the Möbius function of number theory. Show that if n is a positive integer and $n > 1$ then

$$\sum_{j=1}^{n} \mu(j) = - \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} j \sum_{k=\lceil (n+1)/(2j+3) \rceil}^{\lfloor n/(2j+1) \rfloor} \mu(k).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

In the RHS, the range of the integer k is

$$\frac{n+1}{2j+3} \leq k \leq \frac{n}{2j+1},$$

hence

$$\frac{1}{2}\left(\frac{n}{k} - 1\right) - 1 + \frac{1}{2k} = \frac{1}{2}\left(\frac{n+1}{k} - 3\right) \leq j \leq \frac{1}{2}\left(\frac{n}{k} - 1\right).$$

It's easy to see that there is only one integer j that satisfies the above inequalities and therefore, by exchanging the two sums, the RHS can be written as

$$- \sum_{k=1}^{\lfloor (n-1)/3 \rfloor} \mu(k) \left\lfloor \frac{1}{2}\left(\frac{n}{k} - 1\right) \right\rfloor = - \sum_{k=1}^{n} \mu(k) \left\lfloor \frac{1}{2}\left(\frac{n}{k} - 1\right) \right\rfloor$$

so it suffices to prove that

$$\sum_{k=1}^{n} \mu(k) g(n/k) = \sum_{k \geq 1} \mu(k) g(n/k) = f(n) = [n = 1]$$

where $g(x) = 1 + \left\lfloor \frac{1}{2}(x-1) \right\rfloor$ and $f(x) = [\lfloor x \rfloor = 1]$. Note that

$$\sum_{j \geq 1} f(x/j) = \sum_{j \geq 1} \left[\left\lfloor \frac{x}{j} \right\rfloor = 1 \right] = \sum_{j \geq 1} [(x/2) < j \leq x] = 1 + \left\lfloor \frac{1}{2}(x-1) \right\rfloor = g(x),$$

Finally, since $\sum_{k|m} \mu(k) = [m = 1]$ and $\sum_{j \geq 1} \sum_{k \geq 1} |f(n/(kj))| < \infty$

$$\begin{aligned}
\sum_{k \geq 1} \mu(k) g(n/k) &= \sum_{k \geq 1} \mu(k) \sum_{j \geq 1} f(n/(kj)) = \sum_{j \geq 1} \sum_{k \geq 1} \mu(k) f(n/(kj)) \\
&= \sum_{m \geq 1} f(n/m) \sum_{k \geq 1} \mu(k) [m = kj] = \sum_{m \geq 1} f(n/m) \sum_{k|m} \mu(k) \\
&= \sum_{m \geq 1} f(n/m) \sum_{k|m} \mu(k) = \sum_{m \geq 1} f(n/m) [m = 1] = f(n).
\end{aligned}$$

□

Remark. This inversion law for the Möbius function holds for more general f and g:

$$g(x) = \sum_{k \geq 1} f(x/k) \quad \text{if and only if} \quad f(x) = \sum_{k \geq 1} \mu(k) g(x/k)$$

as soon as $\sum_{j \geq 1} \sum_{k \geq 1} |f(n/(kj))| < \infty$ (see for example *Concrete Mathematics* by Graham, Knuth, and Patashnik).