Problem 11344

Proposed by Albert Stadler, Switzerland.

Let \( \mu \) be the Möbius function of number theory. Show that if \( n \) is a positive integer and \( n > 1 \) then
\[
\sum_{j=1}^{n} \mu(j) = - \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} j \sum_{k=\lceil (n+1)/(2j+3) \rceil}^{\lfloor n/(2j+1) \rfloor} \mu(k).
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

In the RHS, the range of the integer \( k \) is
\[
\frac{n+1}{2j+3} \leq k \leq \frac{n}{2j+1},
\]
hence
\[
\frac{1}{2} \left( \frac{n}{k} - 1 \right) - 1 + \frac{1}{2} \left( \frac{n+1}{k} - 3 \right) \leq j \leq \frac{1}{2} \left( \frac{n}{k} - 1 \right).
\]

It’s easy to see that there is only one integer \( j \) that satisfies the above inequalities and therefore, by exchanging the two sums, the RHS can be written as
\[
\sum_{k=1}^{\lfloor (n-1)/3 \rfloor} \mu(k) \left\lfloor \frac{1}{2} \left( \frac{n}{k} - 1 \right) \right\rfloor = - \sum_{k=1}^{n} \mu(k) \left\lfloor \frac{1}{2} \left( \frac{n}{k} - 1 \right) \right\rfloor
\]
so it suffices to prove that
\[
\sum_{k=1}^{n} \mu(k)g(n/k) = \sum_{k \geq 1} \mu(k)g(n/k) = f(n) = [n = 1]
\]
where \( g(x) = 1 + \left\lfloor \frac{1}{2} (x-1) \right\rfloor \) and \( f(x) = \lfloor x \rfloor = 1 \). Note that
\[
\sum_{j \geq 1} f(x/j) = \sum_{j \geq 1} \left\lfloor \frac{x}{j} \right\rfloor = \sum_{j \geq 1} \left( (x/2) < j \leq x \right) = 1 + \left\lfloor \frac{1}{2} (x-1) \right\rfloor = g(x),
\]
Finally, since \( \sum_{k|m} \mu(k) = [m = 1] \) and \( \sum_{j \geq 1} \sum_{k \geq 1} |f(n/(kj))| < \infty \)
\[
\sum_{k \geq 1} \mu(k)g(n/k) = \sum_{k \geq 1} \mu(k) \sum_{j \geq 1} f(n/(kj)) = \sum_{j \geq 1} \sum_{k \geq 1} \mu(k)f(n/(kj)) = \sum_{m \geq 1} f(n/m) \sum_{k \geq 1} \mu(k) \sum_{m \geq 1} f(n/m) \sum_{k \geq 1} \mu(k) = \sum_{m \geq 1} f(n/m) \sum_{k \geq 1} \mu(k) = \sum_{m \geq 1} f(n/m)[m = 1] = f(n).
\]

Remark. This inversion law for the Möbius function holds for more general \( f \) and \( g \):
\[
g(x) = \sum_{k \geq 1} f(x/k) \quad \text{if and only if} \quad f(x) = \sum_{k \geq 1} \mu(k)g(x/k)
\]as soon as \( \sum_{j \geq 1} \sum_{k \geq 1} |f(n/(kj))| < \infty \) (see for example Concrete Mathematics by Graham, Knuth, and Patashnik).