Problem 11343
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Proposed by D. Beckwith (USA).

Show that when n is a positive integer,

$$\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k}=\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}3^{n-2k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that for $|z|<1/4$:

$$\frac{1}{\sqrt{1-4z}}=\sum_{k\geq 0}\binom{2k}{k}z^k \qquad \text{and} \qquad \frac{z^k}{(1-z)^{k+1}}=\sum_{n\geq k}\binom{n}{k}z^n.$$

Now for $|z|<1/5$

$$\begin{aligned}\frac{1}{\sqrt{1-6z+5z^2}}=\frac{1}{1-z}\cdot\frac{1}{\sqrt{1-4(z/(1-z))}} &= \sum_{k\geq 0}\binom{2k}{k}\frac{z^k}{(1-z)^{k+1}}\\ &= \sum_{k\geq 0}\binom{2k}{k}\sum_{n\geq k}\binom{n}{k}z^n\\ &= \sum_{n\geq 0}z^n\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt{1-6z+5z^2}}=\frac{1}{1-3z}\cdot\frac{1}{\sqrt{1-4(z/(1-3z))^2}} &= \sum_{k\geq 0}\binom{2k}{k}3^{-2k}\frac{(3z)^{2k}}{(1-3z)^{2k+1}}\\ &= \sum_{k\geq 0}\binom{2k}{k}3^{-2k}\sum_{n\geq 2k}\binom{n}{2k}(3z)^n\\ &= \sum_{n\geq 0}z^n\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}3^{n-2k}\end{aligned}$$

therefore the identity follows by comparing the two generating functions. □