Problem 11339
(American Mathematical Monthly, Vol.115, January 2008)

Proposed by J. L. Díaz-Barrero (Spain).

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Prove that for all $n \geq 1$,

$$\frac{1}{2}\left(F_n^{1/F_n} + L_n^{1/L_n}\right) \leq 2 - \frac{F_{n+1}}{F_{2n}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is easy to verify it for $n = 1, 2, 3, 4$:

$$\begin{aligned}
n = 1: &\quad \frac{1}{2}\left(F_1^{1/F_1} + L_1^{1/L_1}\right) = 1 \leq 1 = 2 - \frac{F_2}{F_2}\\
n = 2: &\quad \frac{1}{2}\left(F_2^{1/F_2} + L_2^{1/L_2}\right) = \frac{1}{2}\left(1 + \sqrt[3]{3}\right) \leq \frac{4}{3} = 2 - \frac{F_3}{F_4}\\
n = 3: &\quad \frac{1}{2}\left(F_3^{1/F_3} + L_3^{1/L_3}\right) = \frac{1}{2}\left(\sqrt{2} + \sqrt[4]{4}\right) \leq \frac{15}{8} = 2 - \frac{F_4}{F_6}\\
n = 4: &\quad \frac{1}{2}\left(F_4^{1/F_4} + L_4^{1/L_4}\right) = \frac{1}{2}\left(\sqrt[3]{3} + \sqrt[7]{7}\right) \leq \frac{37}{21} = 2 - \frac{F_5}{F_8}
\end{aligned}$$

Since $x^{1/x}$ is a positive decreasing function for $x \geq e$, and $\{F_n\}_{n\geq 4}$ and $\{L_n\}_{n\geq 4}$ are increasing sequences greater than e then

$$\frac{1}{2}\left(F_n^{1/F_n} + L_n^{1/L_n}\right)$$

is a decreasing sequence for $n \geq 4$.
Moreover $2 - (F_{n+1}/F_{2n})$ is an increasing sequence for $n \geq 4$ because

$$2 - (F_{n+1}/F_{2n}) \leq 2 - (F_{n+2}/F_{2n+2})$$

that is equivalent to

$$(F_{n+1} + F_n)F_{2n} = F_{n+2}F_{2n} \leq F_{n+1}F_{2n+2} = F_{n+1}(F_{2n+1} + F_{2n})$$

or $F_nF_{2n} \leq F_{n+1}F_{2n+1}$ which certainly holds since F_n is increasing. Therefore for $n \geq 4$

$$\frac{1}{2}\left(F_n^{1/F_n} + L_n^{1/L_n}\right) \leq \frac{1}{2}\left(F_4^{1/F_4} + L_4^{1/L_4}\right) \leq 2 - \frac{F_5}{F_8} \leq 2 - \frac{F_{n+1}}{F_{2n}}.$$

□