Problem 11338
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Proposed by O. Furdui (Romania).

Let Γ *denote the classical gamma function, and let* $G(n) = \prod_{k=1}^{n} \Gamma(1/k)$. *Find*

$$\lim_{n\to\infty} \left(G(n+1)^{1/(n+1)} - G(n)^{1/n} \right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that

$$\log \Gamma(1/k) = \log(k) - \gamma/k + \sum_{m=1}^{\infty} (1/(km) - \log(1 + 1/(km)))$$

Therefore

$$\log G(n) = \sum_{k=1}^{n} \log \Gamma(1/k) = \log(n!) - \gamma H_n + h(n)$$

where $h(n)$

$$h(n) = \sum_{k=1}^{n} \sum_{m=1}^{\infty} (1/(km) - \log(1 + 1/(km))).$$

Since $0 \leq x - \log(1 + x) \leq x^2$ for $x \geq 0$ then $h(n)$ is uniformly bounded because

$$0 \leq h(n) \leq \sum_{k=1}^{n} \sum_{m=1}^{\infty} 1/(km)^2 \leq (\pi^2/6)^2$$

and $h(n + 1) - h(n) = o(1)$ because

$$0 \leq h(n+1) - h(n) = \sum_{m=1}^{\infty} (1/((n+1)m) - \log(1 + 1/((n+1)m))) \leq (\pi^2/6)/(n+1)^2.$$

Morever

$$n! = (n/e)^n \sqrt{2\pi n}(1 + O(1/n)) \quad \text{and} \quad H_n = \log(n) + \gamma + O(1/n).$$

Therefore

$$a_n := \log(G(n))/n = \log(n) - 1 - (\gamma - 1/2)\log(n)/n + (\log\sqrt{2\pi} - \gamma^2 + h(n))/n + o(1/n).$$

Finally

$$\lim_{n\to\infty} \left(G(n+1)^{1/(n+1)} - G(n)^{1/n} \right) = \lim_{n\to\infty} e^{a_n}(e^{a_{n+1}-a_n} - 1) = \lim_{n\to\infty} (n/e)(1 + o(1))(1/n + o(1/n)) = 1/e$$

because

$$e^{a_n} = e^{\log(n)-1+o(1)} = (n/e)(1 + o(1))$$

and

$$\begin{aligned}
a_{n+1} - a_n &= \log(n+1) - \log(n) - (\gamma - 1/2)(\log(n+1)/(n+1) - \log(n)/n) \\
&\quad + (\log\sqrt{2\pi} - \gamma^2)(1/(n+1) - 1/n) + h(n+1)/(n+1) - h(n)/n + o(1/n) \\
&= 1/n + (\gamma - 1/2)(\log(n) - 1)/n^2 - (\log\sqrt{2\pi} - \gamma^2)/n^2 + (h(n+1) - h(n))/n + o(1/n) \\
&= 1/n + o(1/n).
\end{aligned}$$

□