Problem 11333
(American Mathematical Monthly, Vol.114, December 2007)

Proposed by P. F. Refolio (Spain).

Show that

$$\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right)=\pi.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we consider the finite telescopic product

$$\begin{aligned}\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^{n} &= \frac{3^2}{1^2}\cdot\frac{4^3}{2^3}\cdot\frac{5^4}{3^4}\cdots\frac{N^{N-1}}{(N-2)^{N-1}}\cdot\frac{(N+1)^N}{(N-1)^N}\\ &= \frac{N^{N-1}(N+1)^N}{2((N-1)!)^2}=\frac{N^{2N+1}}{2(N!)^2}\left(1+\frac{1}{N}\right)^{N}\\ &= \frac{e^{2N+1}}{4\pi}(1+o(1))\end{aligned}$$

where in the last step we used

$$N!=\sqrt{2\pi N}(N/e)^N(1+o(1))\quad\text{and}\quad(1+1/N)^N=e(1+o(1)).$$

Now we consider another finite telescopic product

$$\begin{aligned}\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1} &= \prod_{n=2}^{N}\frac{(n-1)^{n^2-1}(n+1)^{n^2-1}}{n^{2(n^2-1)}}\\ &= \frac{1^3\cdot 3^3}{2^6}\cdot\frac{2^8\cdot 4^8}{3^{16}}\cdot\frac{3^{15}\cdot 5^{15}}{4^{30}}\cdots\frac{(N-1)^{N^2-1}\cdot(N+1)^{N^2-1}}{N^{2(N^2-1)}}\\ &= \frac{((N-1)!)^2\cdot(N+1)^{N^2-1}}{N^{N^2+2N-2}}=\frac{((N)!)^2}{N^{2N+1}}\cdot\left(1+\frac{1}{N}\right)^{N^2-1}\\ &= \frac{2\pi}{e^{2N}}\cdot e^{N-\frac{1}{2}}(1+o(1))=\frac{2\pi}{e^{N+\frac{1}{2}}}(1+o(1))\end{aligned}$$

where in the last step we used the Stirling approximation and

$$\left(1+\frac{1}{N}\right)^{N^2-1}=\exp\left((N^2-1)\left(\frac{1}{N}-\frac{1}{2N^2}+o(1/N^2)\right)\right)=\exp\left(N-\frac{1}{2}+o(1)\right)=e^{N-\frac{1}{2}}(1+o(1)).$$

Finally

$$\begin{aligned}\prod_{n=2}^{N}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right) &= \left(\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^2-1}\right)^2\cdot\prod_{n=2}^{N}\left(\frac{n+1}{n-1}\right)^{n}\\ &= \frac{4\pi^2}{e^{2N+1}}(1+o(1))\cdot\frac{e^{2N+1}}{4\pi}(1+o(1))\to\pi.\end{aligned}$$

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