Problem 11333
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Proposed by R. Tauraso (Italy).

Let α and β be positive irrational numbers. Show that for any positive integer n,

$$\sum_{k=0}^{\lfloor n/\alpha\rfloor-1}\left\lfloor\frac{\lceil(k+\{n/\alpha\})\alpha\rceil}{\beta}\right\rfloor=\sum_{k=0}^{\lfloor n/\beta\rfloor-1}\left\lfloor\frac{\lceil(k+\{n/\beta\})\beta\rceil}{\alpha}\right\rfloor$$

where $\{x\}$ denotes the fractional part of x.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\lceil(k+\{n/\alpha\})\alpha\rceil=\lceil(k+n/\alpha-\lfloor n/\alpha\rfloor)\alpha\rceil=\lceil n-j\alpha\rceil=n+\lceil -j\alpha\rceil=n-\lfloor j\alpha\rfloor$$

where $j=\lfloor n/\alpha\rfloor-k$ and therefore

$$\sum_{k=0}^{\lfloor n/\alpha\rfloor-1}\left\lfloor\frac{\lceil(k+\{n/\alpha\})\alpha\rceil}{\beta}\right\rfloor=\sum_{j=1}^{\lfloor n/\alpha\rfloor}\left\lfloor\frac{n-\lfloor j\alpha\rfloor}{\beta}\right\rfloor.$$

Since $(n-\lfloor j\alpha\rfloor)/\beta$ is never an integer then for any positive integer k

$$[\lfloor j\alpha\rfloor+\lfloor k\beta\rfloor<n]=[\lfloor k\beta\rfloor<n-\lfloor j\alpha\rfloor]=[k\beta<n-\lfloor j\alpha\rfloor]=[k<(n-\lfloor j\alpha\rfloor)/\beta]=[k\leq\lfloor(n-\lfloor j\alpha\rfloor)/\beta\rfloor]$$

where $[\cdot]$ is the indicator function. Hence

$$\sum_{k=1}^{\lfloor n/\beta\rfloor}[\lfloor j\alpha\rfloor+\lfloor k\beta\rfloor<n]=\sum_{k=1}^{\lfloor n/\beta\rfloor}[k\leq\lfloor(n-\lfloor j\alpha\rfloor)/\beta\rfloor]=\left\lfloor\frac{n-\lfloor j\alpha\rfloor}{\beta}\right\rfloor.$$

Finally

$$\sum_{k=0}^{\lfloor n/\alpha\rfloor-1}\left\lfloor\frac{\lceil(k+\{n/\alpha\})\alpha\rceil}{\beta}\right\rfloor=\sum_{j=1}^{\lfloor n/\alpha\rfloor}\left\lfloor\frac{n-\lfloor j\alpha\rfloor}{\beta}\right\rfloor=\sum_{j=1}^{\lfloor n/\alpha\rfloor}\sum_{k=1}^{\lfloor n/\beta\rfloor}[\lfloor j\alpha\rfloor+\lfloor k\beta\rfloor<n]$$

which is symmetric with respect to α and β and the statement has been proved. □