Problem 11321

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Prove or disprove: every monic polynomial with rational coefficients and real zeros is the characteristic polynomial of a symmetric matrix with rational entries.

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We will prove that the polynomial with integer coefficients and real zeros

$$x^2 - 4x + 1$$

is not the characteristic polynomial of a symmetric matrix with rational entries.

Assume that $x^2 - 4x + 1$ is the characteristic polynomial of the symmetric matrix with rational entries

$$\begin{bmatrix}
    a & b \\
    b & c
\end{bmatrix}$$

that is

$$\begin{cases}
    a + c = 4 \\
    ac - b^2 = 1
\end{cases}.$$ 

Hence $c = 4 - a$, $ac - b^2 = a(4 - a) - b^2 = 4 - (a - 2)^2 - b^2 = 1$ and

$$(a - 2)^2 + b^2 = 3.$$ 

Since the matrix has rational entries then $a - 2 = p_1/q_1$ and $b = p_2/q_2$ for some integers $p_1$, $p_2$, $q_1$ and $q_2$. Therefore

$$p_1^2 + p_2^2 = 3(q_1q_2)^2,$$

which means in the factorization of positive integer $p_1^2 + p_2^2$, the exponent of 3 is odd. This is a contradiction because a positive integer is the sum of two squares if and only if all the prime factors congruent to 3 mod 4 have an even exponent in its prime-factorization.