Problem 11321

Proposed by C. Hillar (USA).

Prove or disprove: every monic polynomial with rational coefficients and real zeros is the characteristic polynomial of a symmetric matrix with rational entries.

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We will prove that the polynomial with integer coefficients and real zeros
\[ x^2 - 4x + 1 \]

is not the characteristic polynomial of a symmetric matrix with rational entries.

Assume that \( x^2 - 4x + 1 \) is the characteristic polynomial of the symmetric matrix with rational entries

\[
\begin{bmatrix}
  a & b \\
  b & c \\
\end{bmatrix}
\]

that is
\[
\begin{cases}
  a + c = 4 \\
  ac - b^2 = 1
\end{cases}
\]

Hence \( c = 4 - a \), \( ac - b^2 = a(4 - a) - b^2 = 4 - (a - 2)^2 - b^2 = 1 \) and \( (a - 2)^2 + b^2 = 3 \).

Since the matrix has rational entries then \( a - 2 = p_1/q_1 \) and \( b = p_2/q_2 \) for some integers \( p_1, p_2, q_1 \) and \( q_2 \). Therefore
\[ p_1^2 + p_2^2 = 3(q_1q_2)^2, \]
which means in the factorization of positive integer \( p_1^2 + p_2^2 \), the exponent of 3 is odd. This is a contradiction because a positive integer is the sum of two squares if and only if all the prime factors congruent to 3 mod 4 have an even exponent in its prime-factorization.

\[ \square \]