Problem 11321
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Proposed by C. Hillar (USA).

Prove or disprove: every monic polynomial with rational coefficients and real zeros is the characteristic polynomial of a symmetric matrix with rational entries.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that the polynomial with integer coefficients and real zeros

$$x^2 - 4x + 1$$

is not the characteristic polynomial of a symmetric matrix with rational entries.

Assume that $x^2 - 4x + 1$ is the characteristic polynomial of the symmetric matrix with rational entries

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

that is

$$\begin{cases} a + c = 4 \\ ac - b^2 = 1 \end{cases}.$$

Hence $c = 4 - a$, $ac - b^2 = a(4 - a) - b^2 = 4 - (a - 2)^2 - b^2 = 1$ and

$$(a - 2)^2 + b^2 = 3.$$

Since the matrix has rational entries then $a - 2 = p_1/q_1$ and $b = p_2/q_2$ for some integers p_1, p_2, q_1 and q_2. Therefore

$$p_1^2 + p_2^2 = 3(q_1 q_2)^2,$$

which means in the factorization of positive integer $p_1^2 + p_2^2$, the exponent of 3 is odd. This is a contradiction because a positive integer is the sum of two squares if and only if all the prime factors congruent to 3 mod 4 have an even exponent in its prime-factorization. $\square$