Problem 11319
(American Mathematical Monthly, Vol.114, November 2007)

Proposed by D. Beckwith (USA).

Let q be an integer greater than 1. For $n \geq 1$, let Φ_n be the polynomial function on the complex numbers given by $\Phi_n(z) = \sum_{j=0}^{n-1} z^j$. Let $S(k)$ denote the sum of the digits in the base q representation of k. Show that for $|z| < 1$,

$$\prod_{n=1}^{\infty} \Phi_q\left(z^{\Phi_n(q)}\right) = \sum_{m=0}^{\infty} z^{(qm-S(m))/(q-1)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Note that

$$\prod_{n=1}^{\infty} \Phi_q\left(z^{\Phi_n(q)}\right) = \prod_{n=1}^{\infty} \left(z^{0\Phi_n(q)} + z^{1\Phi_n(q)} + \cdots + z^{(q-1)\Phi_n(q)}\right) = \sum_{S} z^{\sum_{j=1}^{\infty} c_j \Phi_j(q)}$$

where the last sum is extended over all sequences $\{c_j\}_{j\geq 0}$ that are eventually 0 and such that c_j is a q-digit that is $c_j \in \{0, 1, \ldots, q-1\}$.
Let $\sum_{j=1}^{\infty} c_j q^{j-1}$ the q-base representation of a non negative number m then define the function Ψ such that

$$\Psi(m) = \sum_{j=1}^{\infty} c_j \Phi_j(q) = \sum_{j=1}^{\infty} c_j \left(\frac{q^j - 1}{q-1}\right) = \frac{1}{q-1}\left(q\sum_{j=1}^{\infty} c_j q^{j-1} - \sum_{j=1}^{\infty} c_j\right) = \frac{qm - S(m)}{q-1}.$$

Therefore

$$\prod_{n=1}^{\infty} \Phi_q\left(z^{\Phi_n(q)}\right) = \sum_{S} z^{\sum_{j=1}^{\infty} c_j \Phi_j(q)} = \sum_{m=0}^{\infty} z^{\psi(m)} = \sum_{m=0}^{\infty} z^{(qm-S(m))/(q-1)}.$$

The function Ψ is strictly increasing: $\Psi(m+1) > \Psi(m)$ iff $q > S(m+1) - S(m)$ which holds because $1 \leq S(m+1) - S(m)$. Hence the coefficients of the final power series are 0 or 1 and they are not eventually 0. This implies that the radius of convergence is 1 and the series converges for $|z| < 1$.$\square$