Problem 11313
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Let f be a four-times differentiable function on $\mathbb{R}$ with $f^{(4)}$ continuous on $[0,1]$ such that

$$\int_0^1 f(x)\,dx + 3f(1/2) = 8\int_{1/4}^{3/4} f(x)\,dx.$$

Prove that there is some $c \in (0,1)$ such that $f^{(4)}(c) = 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Letting $g(x) = f(1/2+x) - f(1/2)$ and

$$G(t) = \int_{-t}^{t} g(x)\,dx - 8\int_{-t/2}^{t/2} g(x)\,dx$$

then $G(0) = 0$ and the required condition is equivalent to $G(1/2) = 0$.
Hence, by the Mean Value Theorem there is $t_0 \in (0, 1/2)$ such that $G'(t_0) = 0$. Since

$$G'(t) = g(t) - 4g(t/2) - 4g(-t/2) + g(-t)$$

and $G'(0) = G'(t_0) = 0$ (note that $g(0) = 0$) then, again by the Mean Value Theorem, there is $t_1 \in (0, t_0)$ such that $G''(t_1) = 0$. Since

$$G''(t) = g'(t) - 2g'(t/2) + 2g'(-t/2) - g'(-t)$$

and $G''(0) = G''(t_1) = 0$ then, again by the Mean Value Theorem, there is $t_2 \in (0, t_1)$ such that $G'''(t_2) = 0$. Since

$$G'''(t) = (g''(t) - g''(t/2)) - (g''(-t/2) - g''(-t))$$

then by the Mean Value Theorem there are $\theta_+ \in (t_2/2, t_2)$ and $\theta_- \in (-t_2, -t_2/2)$ such that

$$0 = G'''(t_2) = g'''(\theta_+)t/2 - g'''(\theta_-)t_2/2.$$

By applying one more time the Mean Value Theorem there is $\theta \in (\theta_-, \theta_+) \subset (-1/2, 1/2)$ such that

$$0 = g^{(4)}(\theta)(\theta_+ - \theta-)t/2$$

that is, letting $c = 1/2 + \theta \in (0,1)$,

$$f^{(4)}(c) = g^{(4)}(\theta) = 0.$$

□