Problem 11306
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Let a, b, and c be the lenghts of the sides of a nondegenerate triangle, let $p = (a+b+c)/2$, and let r and R be the inradius and circumradius of the triangle, respectively. Show that

$$\frac{a}{2} \cdot \frac{4r-R}{R} \leq \sqrt{(p-b)(p-c)} \leq \frac{a}{2}$$

and determine the cases of equality.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since

$$r = \frac{A}{p},\ R = \frac{abc}{4A},\ A = \sqrt{p(p-a)(p-b)(p-c)}$$

then

$$\frac{4r-R}{2R} = 2 \cdot \frac{A}{p} \cdot \frac{4A}{abc} - \frac{1}{2} = \frac{8(p-a)(p-b)(p-c)}{abc} - \frac{1}{2}.$$

Moreover letting

$$x = b+c-a = 2(p-a),\ y = c+a-b = 2(p-b),\ z = a+b-c = 2(p-c)$$

the inequalities become

$$\frac{8xyz}{(x+y)(y+z)(x+z)} - \frac{1}{2} \leq \frac{\sqrt{y}\sqrt{z}}{y+z} \leq \frac{1}{2}.$$

Note that since the triangle is nondegenerate then x,y,z are positive real numbers.
The first inequality is equivalent to

$$\frac{8xyz}{(x+y)(y+z)(x+z)} \leq \frac{1}{2} + \frac{\sqrt{y}\sqrt{z}}{y+z} = \frac{(\sqrt{y}+\sqrt{z})^2}{2(y+z)}$$

that is

$$xyz \leq \frac{x+y}{2} \cdot \frac{x+z}{2} \cdot \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2$$

which holds because

$$\sqrt{xy} \leq \frac{x+y}{2},\ \sqrt{xz} \leq \frac{x+z}{2},\ \sqrt{yz} \leq \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2$$

by AGM inequality. Equality holds iff $x = y = z$ that is when $a = b = c$.
The second inequality is equivalent to

$$\sqrt{yz} \leq \frac{y+z}{2}$$

which holds by AGM inequality. Equality holds iff $y = z$ that is when $b = c$. □