Problem 11302
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Proposed by H. Alzer (Germany).

Find

$$\sum_{k=2}^{\infty} \frac{(2k+1)H_k^2}{(k-1)k(k+1)(k+2)}$$

where H_k is the kth harmonic number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\sum_{k=2}^{\infty} \frac{(2k+1)H_k^2}{(k-1)k(k+1)(k+2)} = \frac{1}{2}\left(\sum_{k=2}^{\infty} H_k^2 \Delta\left(\frac{1}{k+1}\right) - \sum_{k=2}^{\infty} H_k^2 \Delta\left(\frac{1}{k-1}\right)\right).$$

By summation by parts

$$\begin{aligned}
\sum_{k=2}^{\infty} H_k^2 \Delta\left(\frac{1}{k+1}\right) &= \left[\frac{H_k^2}{k+1}\right]_2^{\infty} - \sum_{k=2}^{\infty} \frac{1}{k+2}\Delta\left(H_k^2\right) \\
&= -\frac{3}{4} - \sum_{k=2}^{\infty} \frac{1}{k+2}\left(\left(H_k + \frac{1}{k+1}\right)^2 - H_k^2\right) \\
&= -\frac{3}{4} - 2\sum_{k=2}^{\infty} \frac{H_k}{(k+1)(k+2)} - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2(k+2)} \\
&= -\frac{3}{4} + 2\sum_{k=2}^{\infty} H_k \Delta\left(\frac{1}{k+1}\right) - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2(k+2)} \\
&= -\frac{3}{4} + 2\left[\frac{H_k}{k+1}\right]_2^{\infty} - 2\sum_{k=2}^{\infty} \frac{1}{k+2}\Delta\left(H_k\right) - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2(k+2)} \\
&= -\frac{3}{4} - 1 - 2\sum_{k=2}^{\infty} \frac{1}{(k+1)(k+2)} - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2(k+2)} \\
&= -\frac{7}{4} - \sum_{k=2}^{\infty} \frac{1}{(k+1)(k+2)} - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2} \\
&= -\frac{7}{4} - \sum_{k=3}^{\infty} \frac{1}{k(k+1)} - \sum_{k=3}^{\infty} \frac{1}{k^2}
\end{aligned}$$

In a similar way

$$\begin{aligned}
\sum_{k=2}^{\infty} H_k^2 \Delta\left(\frac{1}{k-1}\right) &= \left[\frac{H_k^2}{k-1}\right]_2^{\infty} - \sum_{k=2}^{\infty} \frac{1}{k} \Delta\left(H_k^2\right) \\
&= -\frac{9}{4} - \sum_{k=2}^{\infty} \frac{1}{k}\left(\left(H_k + \frac{1}{k+1}\right)^2 - H_k^2\right) \\
&= -\frac{9}{4} - 2\sum_{k=2}^{\infty} \frac{H_k}{k(k+1)} - \sum_{k=2}^{\infty} \frac{1}{k(k+1)^2} \\
&= -\frac{9}{4} + 2\sum_{k=2}^{\infty} H_k \Delta\left(\frac{1}{k}\right) - \sum_{k=2}^{\infty} \frac{1}{k(k+1)^2} \\
&= -\frac{9}{4} + 2\left[\frac{H_k}{k}\right]_2^{\infty} - 2\sum_{k=2}^{\infty} \frac{1}{k+1} \Delta\left(H_k\right) - \sum_{k=2}^{\infty} \frac{1}{k(k+1)^2} \\
&= -\frac{9}{4} - \frac{3}{2} - 2\sum_{k=2}^{\infty} \frac{1}{(k+1)^2} - \sum_{k=2}^{\infty} \frac{1}{k(k+1)^2} \\
&= -\frac{15}{4} - \sum_{k=2}^{\infty} \frac{1}{k(k+1)} - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2} \\
&= -\frac{15}{4} - \sum_{k=2}^{\infty} \frac{1}{k(k+1)} - \sum_{k=3}^{\infty} \frac{1}{k^2}
\end{aligned}$$

Finally

$$\sum_{k=2}^{\infty} \frac{(2k+1)H_k^2}{(k-1)k(k+1)(k+2)} = \frac{1}{2}\left(-\frac{7}{4} - \sum_{k=3}^{\infty} \frac{1}{k(k+1)} - \sum_{k=3}^{\infty} \frac{1}{k^2} + \frac{15}{4} + \sum_{k=2}^{\infty} \frac{1}{k(k+1)} + \sum_{k=3}^{\infty} \frac{1}{k^2}\right) = \frac{13}{12}.$$

□