Problem 11299
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Show that

$$\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right)=\frac{e\sqrt{e}}{2\pi}.$$

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Taking the logarithm the infinite product (which converges) becomes

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(-1+(n^2-1)\log\left(\frac{1}{1-1/n^2}\right)\right) &= \sum_{n=2}^{\infty}\left(-1+(n^2-1)\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}\right)\\
&= \sum_{n=2}^{\infty}\left(-1+\sum_{k=1}^{\infty}\frac{1}{kn^{2k-2}}-\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}\right)\\
&= \sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{(k+1)n^{2k}}-\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}\right)\\
&= \sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k+1}-\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k}
\end{aligned}$$

Now the second series yields

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k} &= \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}=\sum_{n=2}^{\infty}\log\left(\frac{n^2}{n^2-1}\right)\\
&= \lim_{N\to\infty}\log\left(\prod_{n=2}^{N}\frac{n^2}{(n-1)(n+1)}\right)=\lim_{N\to\infty}\log\left(\frac{2N}{N+1}\right)=\log 2.
\end{aligned}$$

So it suffices to prove that

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k+1}=\frac{3}{2}-\log\pi$$

which is interesting by itself. Since the *cotangent identity*

$$\cot x=\sum_{n=0}^{\infty}\frac{1}{x-n\pi}.$$

holds in the interval $0<x<\pi$, then

$$\begin{aligned}
x^2\cot x-x+\frac{x^2}{x-\pi}-\frac{x^2}{x+\pi} &= 2\sum_{n=2}^{\infty}\frac{x}{x^2-(n\pi)^2}=-2\sum_{n=2}^{\infty}\frac{x^3/\pi^2}{1-(x/(n\pi))^2}\\
&= -2x\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2k}=-2\sum_{k=1}^{\infty}\frac{x^{2k+1}}{\pi^{2k}}(\zeta(2k)-1).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k+1} &= \frac{2}{\pi^2}\int_0^{\pi}\sum_{k=1}^{\infty}\frac{x^{2k+1}}{\pi^{2k}}(\zeta(2k)-1)\,dx \\
&= -\frac{1}{\pi^2}\int_0^{\pi}\left(x^2\cot\,x - x - \frac{x^2}{x-\pi} - \frac{x^2}{x+\pi}\right)\,dx \\
&= \frac{1}{\pi^2}\int_0^{\pi}\left(x + \frac{x^2}{x+\pi}\right)\,dx + \frac{1}{\pi^2}\lim_{t\to\pi^-}\left(\int_0^t \frac{x^2}{x-\pi}\,dx - \int_0^t x^2\cot\,x\,dx\right) \\
&= \log 2 + \frac{1}{\pi^2}\lim_{t\to\pi^-}\left(\left[\frac{x^2}{2}+\pi x+\pi^2\log(\pi-x)\right]_0^t - \pi^2\log(\pi-t)-\pi^2\log 2+o(1)\right) \\
&= \frac{3}{2}-\log\pi
\end{aligned}$$

where we used the following asymptotic estimate as $t\to\pi^-$

$$\begin{aligned}
\int_0^t x^2\cot\,x\,dx &= \int_0^t x^2\,d(\log(\sin x)) = \left[x^2\log(\sin x)\right]_0^t - 2\int_0^t x\,\log(\sin x)\,dx \\
&= \pi^2\log((\pi-t)(1+o(1))) - 2\int_0^t x\,\log(\sin x)\,dx \\
&= \pi^2\log(\pi-t)+\pi^2\log 2+o(1)
\end{aligned}$$

and

$$\int_0^{\pi} x\,\log(\sin x)\,dx = -\frac{\pi^2}{2}\,\log 2.$$

The last integral can be evaluated in the following way

$$\int_0^{\pi} x\,\log(\sin x)\,dx = \int_0^{\pi}(\pi-x)\log(\sin(\pi-x))\,dx = \pi\int_0^{\pi}\log(\sin x)\,dx - \int_0^{\pi} x\log(\sin x)\,dx$$

and therefore

$$\int_0^{\pi} x\,\log(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi}\log(\sin x)\,dx = -\frac{\pi^2}{2}\,\log 2$$

because

$$\begin{aligned}
\int_0^{\pi}\log(\sin x)\,dx &= 2\int_0^{\pi/2}\log(\sin x)\,dx \\
&= 2\int_0^{\pi/2}\log(\sin(\pi/2-x))\,dx = 2\int_0^{\pi/2}\log(\cos x)\,dx \\
&= \int_0^{\pi/2}\log(\sin(2x)/2)\,dx = \frac{1}{2}\int_0^{\pi}\log((\sin x)/2)\,dx \\
&= \frac{1}{2}\int_0^{\pi}\log(\sin x)\,dx - \frac{\pi}{2}\log 2.
\end{aligned}$$

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