Problem 11298
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Show that for $n \geq 3$, *if a convex* n*-gon admits a triangulation in which every vertex is incident with an odd number of triangles then* n *must be a multiple of* 3.

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A vertex is incident with an odd number of triangles if and only if the degree of the vertex is even. Hence every vertex is incident with an odd number of triangles if and only if the triangulation, which is a plane graph, is eulerian or in other words its faces can be colored with two colors, say black and white (it is two-colorable).
We first prove by induction that if n is a multiple of 3 then a convex n-gon admits a two-colorable triangulation. For $n = 3$ it is true, and for $n = 3k > 3$ take a two-colorable triangulation of a convex $(3k - 3)$-gon and assume that the external face is white. Then take a triangulated 5-gon with the central triangle white and the other two triangles black and glue together one of the sides of the $(3k - 3)$-gon and the external side of the white triangle. The final polygon (made it convex) is two-colorable and has $(3k - 3) + 5 - 2 = 3k = n$ vertices.
Now we prove by induction that if n is not a multiple of 3 then a convex n-gon does not admit a two-colorable triangulation. For $n = 4$ and $n = 5$ this is evidently true. Let $n > 6$ be not a multiple of 3 and take a triangulation of a convex n-gon. Assume by contradiction that it is two-colorable and that the external face is white.
We have two cases: all the triangles which are adjacent to the external face are ears or not (an ear of a polygon is a triangle formed by three consecutive vertices). In the first case n is even and after removing all these ears, which are all black, we obtain a two-colorable triangulation of a $(n/2)$-gon (paint the external face black). This would contradict the inductive hypothesis because $3 < n/2 < n$ and $n/2$ is not a multiple of 3.
In the second case, there is a triangle which is adjacent to the external face that is not a ear. This triangle is black and it has two sides that are diagonals. Therefore it splits the n-gon triangulation in two triangulated convex polygons: a n_1-gon and a n_2-gon with $n_1 + n_2 = n + 1$ and $n_1, n_2 \geq 3$. The n_1-gon plus the black triangle is a two-colorable $(n_1 + 1)$-gon triangulation and the n_2-gon plus the black triangle is a two-colorable $(n_2 + 1)$-gon triangulation. Now $3 < n_1 + 1 < n$ and $3 < n_2 + 1 < n$ and by inductive hypothesis they must be multiples of 3. Hence it follows that also $n = n_1 + n_2 - 1 = (n_1 + 1) + (n_2 + 1) - 3$ is a multiple of 3 and this is a contradiction. $\square$