Problem 11290
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Proposed by C. Lupu and T. Lupu (Romania).

Let f and g be continuous real valued functions on $[0,1]$. Prove that there exists $c \in (0,1)$ such that

$$\int_0^1 f(x)\,dx \int_0^c xg(x)\,dx = \int_0^1 g(x)\,dx \int_0^c xf(x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Letting

$$h(t) = g(t)\int_0^1 f(x)\,dx - f(t)\int_0^1 g(x)\,dx$$

it suffices to prove that there exists $c \in (0,1)$ such that

$$\int_0^c th(t)\,dt = \int_0^c tg(t)\,dt \int_0^1 f(x)\,dx - \int_0^c tf(t)\,dt \int_0^1 g(x)\,dx = 0.$$

Let

$$u(s) = \int_0^s H(x)\,dx = \int_{x=0}^s \int_{t=0}^x h(t)\,dt$$

then u is a continuous and differentiable function in $[0,1]$ such that

$$u'(0) = H(0) = \int_0^0 h(t)\,dt = 0$$

and

$$u'(1) = H(1) = \int_0^1 h(t)\,dt = \int_0^1 g(t)\,dt \int_0^1 f(x)\,dx - \int_0^1 f(t)\,dt \int_0^1 g(x)\,dx = 0$$

and by Flett's Theorem (see *A mean value theorem*, Math. Gaz., 42 (1958), 38–39) there exists $c \in (0,1)$ such that

$$u'(c) = H(c) = \frac{u(c) - u(0)}{c - 0} = \frac{u(c)}{c}.$$

Hence

$$\int_0^c th(t)\,dt = \int_0^c t\,d(H(t)) = [tH(t)]_0^c - \int_0^c H(t)\,dt = cH(c) - u(c) = 0.$$

$\square$