Problem 11290

Proposed by C. Lupu and T. Lupu (Romania).

Let \( f \) and \( g \) be continuous real valued functions on \([0, 1]\). Prove that there exists \( c \in (0, 1) \) such that
\[
\int_0^1 f(x) \, dx \int_0^c xg(x) \, dx = \int_0^1 g(x) \, dx \int_0^c xf(x) \, dx.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Universit`a di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Letting \( h(t) = g(t) \int_0^1 f(x) \, dx - f(t) \int_0^1 g(x) \, dx \)

it suffices to prove that there exists \( c \in (0, 1) \) such that
\[
\int_0^c t h(t) \, dt = \int_0^c tg(t) \, dt \int_0^1 f(x) \, dx - \int_0^c t f(t) \, dt \int_0^1 g(x) \, dx = 0.
\]

Let
\[
u(s) = \int_0^s H(x) \, dx = \int_{x=0}^s \int_{t=0}^x h(t) \, dt
\]

then \( u \) is a continuous and differentiable function in \([0, 1]\) such that
\[u'(0) = H(0) = \int_0^0 h(t) \, dt = 0\]

and
\[u'(1) = H(1) = \int_0^1 h(t) \, dt = \int_0^1 g(t) \, dt \int_0^1 f(x) \, dx - \int_0^1 f(t) \, dt \int_0^1 g(x) \, dx = 0\]

and by Flett’s Theorem (see A mean value theorem, Math. Gaz., 42 (1958), 38–39) there exists \( c \in (0, 1) \) such that
\[u'(c) = H(c) = \frac{u(c) - u(0)}{c - 0} = \frac{u(c)}{c}.
\]

Hence
\[
\int_0^c t h(t) \, dt = \int_0^c t d(H(t)) = [tH(t)]_0^c - \int_0^c H(t) \, dt = cH(c) - u(c) = 0.
\]
\[\square\]