Problem 11277
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Proposed by Prithwijit De (Ireland).

Find

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\frac{\log(2-\sin\theta\cos\phi)\sin\theta}{2-2\sin\theta\cos\phi+\sin^2\theta\cos^2\phi}\,d\theta\,d\phi.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that

$$I=\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\frac{\log(2-\sin\theta\cos\phi)}{1+(1-\sin\theta\cos\phi)^2}\cdot\sin\theta\,d\theta\,d\phi=\frac{\pi^2\log(2)}{16}.$$

This is a surface integral over the first octant of the unit sphere with respect to the spherical coordinates. The same integral with respect to the cartesian coordinates is

$$I=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\frac{\log(2-x)}{1+(1-x)^2}\cdot\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx.$$

After performing the improper integral with respect to y we obtain

$$\int_{x=0}^{1}\frac{\log(2-x)}{1+(1-x)^2}\left[\arctan\left(\frac{y}{\sqrt{1-x^2-y^2}}\right)\right]_{y=0}^{\sqrt{1-x^2}}dx=\frac{\pi}{2}\int_0^1\frac{\log(2-x)}{1+(1-x)^2}\,dx=\frac{\pi}{2}\int_0^1\frac{\log(1+x)}{1+x^2}\,dx$$

The Catalan's constant $K\approx 0.9159655942$ has several integral representation. Some of them are the following ones:

$$K=-\int_0^1\frac{\log(x)}{1+x^2}\,dx=-\int_0^1\frac{\log((1-x^2)/2)}{1+x^2}\,dx=-\int_0^1\frac{\log((1-x)/\sqrt{2})}{1+x^2}\,dx.$$

Therefore

$$K=\frac{\pi\log(2)}{4}-\int_0^1\frac{\log(1-x)}{1+x^2}\,dx-\int_0^1\frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi\log(2)}{8}-\int_0^1\frac{\log(1-x)}{1+x^2}\,dx$$

and we find that

$$\int_0^1\frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi\log(2)}{4}-\frac{\pi\log(2)}{8}=\frac{\pi\log(2)}{8}.$$

Finally

$$I=\frac{\pi}{2}\int_0^1\frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi}{2}\cdot\frac{\pi\log(2)}{8}=\frac{\pi^2\log(2)}{16}.$$

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