Problem 11275
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Proposed by M. Becker (USA).

Find

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty}\frac{(x-y)^2\log((x+y)/(x-y))}{xy\sinh(x+y)}\,dx\,dy.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We change the variables: letting $u=x+y$ and $v=(x-y)/(x+y)$, the integral becomes

$$I=\int_{v=0}^{1}\int_{u=0}^{\infty}\frac{-(vu)^2\log(v)}{(u^2(1-v^2)/4)\sinh(u)}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\left(\int_{v=0}^{1}\frac{-2v^2\log(v)}{1-v^2}\,dv\right)\cdot\left(\int_{u=0}^{\infty}\frac{u}{\sinh(u)}\,du\right).$$

The first integral:

$$\int_{v=0}^{1}\frac{-2v^2\log(v)}{1-v^2}\,dv=2\int_{v=0}^{1}\log(v)\,dv-2\int_{v=0}^{1}\frac{\log(v)}{1-v^2}\,dv=-2-2\int_{v=0}^{1}\frac{\log(v)}{1-v^2}\,dv.$$

The second integral: let $t=e^{-u}$ then

$$\int_{u=0}^{\infty}\frac{u}{\sinh(u)}\,du=-2\int_{t=1}^{0}\frac{\log(t)}{t^{-1}-t}\cdot\left(-\frac{1}{t}\right)\,dt=-2\int_{t=0}^{1}\frac{\log(t)}{1-t^2}\,dt.$$

Finally we find

$$\begin{aligned}2\int_0^1\frac{\log(t)}{1-t^2}\,dt &= \int_0^1\frac{\log(t)}{1-t}\,dt+\int_0^1\frac{\log(t)}{1+t}\,dt=-\mathrm{Dilog}(0)+\int_0^1\log(t)\,d(\log(1+t))\\ &= -\mathrm{Dilog}(0)+[\log(t)\log(1+t)]_0^1-\int_0^1\frac{\log(1+t)}{t}\,dt\\ &= -\mathrm{Dilog}(0)+\mathrm{Dilog}(2)=-\frac{\pi^2}{6}-\frac{\pi^2}{12}=-\frac{\pi^2}{4}\end{aligned}$$

where

$$\mathrm{Dilog}(x):=\int_1^x\frac{\log(t)}{1-t}\,dt=\int_0^{1-x}\frac{\log(1/(1-s))}{s}\,ds$$

and

$$\mathrm{Dilog}(0):=\int_0^1\frac{1}{s}\sum_{k=1}^{\infty}\frac{s^k}{k}\,ds=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\quad\text{and}\quad\mathrm{Dilog}(2):=\int_0^{-1}\frac{1}{s}\sum_{k=1}^{\infty}\frac{s^k}{k}\,ds=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=-\frac{\pi^2}{12}$$

Hence

$$I=\left(-2+\frac{\pi^2}{4}\right)\cdot\frac{\pi^2}{4}=\frac{\pi^2(\pi^2-8)}{16}.$$

□