Problem 11274
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Proposed by D. Knuth (USA).

Prove that for nonnegative integers m and n

$$\sum_{k=0}^{m} 2^k \binom{2m-k}{m+n} = 4^m - \sum_{j=1}^{n} \binom{2m+1}{m+j}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

$$\begin{aligned}\sum_{k=0}^{m} 2^k \binom{2m-k}{m+n} &= \sum_{k=0}^{m-n} 2^k \binom{2m-k}{m+n} = \sum_{k=0}^{m-n} \left(\sum_{j=0}^{k} \binom{k}{j}\right) \binom{2m-k}{m+n} = \sum_{k=0}^{m-n} \sum_{j=0}^{k} \binom{2m-k}{m+n}\binom{k}{j} \\ &= \sum_{j=0}^{m-n} \sum_{k=j}^{m-n} \binom{2m-k}{m+n}\binom{k}{j} = \sum_{j=0}^{m-n} \sum_{k=0}^{m-n-j} \binom{2m-j-k}{m+n}\binom{j+k}{j} \\ &= \sum_{j=0}^{m-n} \binom{2m+1}{m+n+j+1} = \sum_{j=n+1}^{m+1} \binom{2m+1}{m+j}\end{aligned}$$

where we used the binomial identity (see *Concrete Mathematics* by Graham, Knuth and Patashnik)

$$\sum_{k=0}^{l-s} \binom{l-k}{s}\binom{q+k}{t} = \binom{l+q+1}{s+t+1}$$

for $s = m+n$, $t = q = j$, and $l = 2m-j$. Therefore

$$\sum_{k=0}^{m} 2^k \binom{2m-k}{m+n} + \sum_{j=1}^{n} \binom{2m+1}{m+j} = \sum_{j=1}^{m+1} \binom{2m+1}{m+j} = \sum_{j=m+1}^{2m+1} \binom{2m+1}{j} = \frac{2^{2m+1}}{2} = 4^m.$$

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