Problem 11270
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Proposed by S. Sadov (Canada).

Let S_n be the $n \times n$ matrix in which the entries are 1 through n^2, spiraling inward with 1 in the $(1,1)$ entry. Show that for $n \geq 2$,

$$\det(S_n) = (-1)^{n(n-1)/2} 4^{n-1} \frac{3n-1}{2} \prod_{k=0}^{n-2} \left(\frac{1}{2} + k \right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$S_n(x) = \begin{pmatrix} x & x+1 & \cdots & x+n-2 & x+n-1 \\ x+4n-5 & x+4n-5 & \cdots & x+5n-7 & x+n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x+3n-3 & x+3n-4 & \cdots & x+2n-1 & x+2n-2 \end{pmatrix}.$$

By adding the last row to the first row of $S_n(x)$ we obtain

$$\begin{pmatrix} 2x+3n-3 & 2x+3n-3 & \cdots & 2x+3n-3 & 2x+3n-3 \\ x+4n-5 & x+4n-5 & \cdots & x+5n-7 & x+n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x+3n-3 & x+3n-4 & \cdots & x+2n-1 & x+2n-2 \end{pmatrix}.$$

and therefore for $n \geq 2$

$$\det(S_n(x)) = (2x+3n-3) \cdot \det(U_n(x))$$

where

$$U_n(x) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ x+4n-5 & x+4n-5 & \cdots & x+5n-7 & x+n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x+3n-3 & x+3n-4 & \cdots & x+2n-1 & x+2n-2 \end{pmatrix}.$$

By subtracting x times the first row from the other rows of $U_n(x)$ we obtain $U_n(0)$ and therefore $\det(U_n(x))$ is independent of x.
By adding the second and the last row to the first row of $U_n(0)$ we obtain

$$\begin{pmatrix} 7n-7 & 7n-7 & \cdots & 7n-7 & 3n-1 \\ 4n-5 & 4n-5 & \cdots & 5n-7 & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 3n-3 & 3n-4 & \cdots & 2n-1 & 2n-2 \end{pmatrix}.$$

Since $3n-1 = (7n-7) - (4n-6)$ then by the linearity of the determinant in the last column we get

$$\det(U_n(0)) = (7n-7) \cdot \det(U_{n-1}(0)) - (-1)^{n+1}(4n-6) \cdot \det(S_{n-1}(2n-1))$$

that is

$$\begin{aligned} (8-7n)\det(U_n(0)) &= (-1)^n(4n-6) \cdot (2(2n-1) + 3(n-1) - 3) \cdot \det(U_{n-1}(0)) \\ &= (-1)^n(4n-6) \cdot (7n-8) \cdot \det(U_{n-1}(0)) \end{aligned}$$

and

$$\begin{aligned}
\det(U_n(0)) &= (-1)^{n-1} 2(2n-3)\cdot \det(U_{n-1}(0)) \\
&= (-1)^{(n-1)+(n-2)} 2^2 (2n-3)(2n-5)\cdot \det(U_{n-2}(0)) \\
&= (-1)^{n(n-1)/2} 2^{n-2} (2n-3)!!
\end{aligned}$$

because $\det(U_2(0)) = -1$. Finally

$$\begin{aligned}
\det(S_n(1)) &= (3n-1)\cdot \det(U_n(0)) \\
&= (-1)^{n(n-1)/2} 2^{n-2} (3n-1)(2n-3)!! \\
&= (-1)^{n(n-1)/2} 2^{n-2} (3n-1) \prod_{k=0}^{n-2} (2k+1) \\
&= (-1)^{n(n-1)/2} 4^{n-1} \frac{3n-1}{2} \prod_{k=0}^{n-2} \left(k+\frac{1}{2}\right).
\end{aligned}$$

$\square$