Problem 11262
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Proposed by A. Burungale (India).

In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person amongst the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The problem is equivalent to the following one.

Given a graph with $2n+1$ vertices, such that for any set A of n vertices there exists a vertex not in A connected to all of them, prove that there exists a vertex connected to all other $2n$ vertices of the graph.

We first prove by induction that there is a $(n+1)$-clique . A point is a 0-clique. If we have a k-clique with $k \le n$ then by hypothesis there is a vertex outside the clique which is connected to all the vertices in the clique. Thus the k-clique and this new vertex form a $(k+1)$-clique.

Now that we have a $(n+1)$-clique, we choose the n vertices of the graph that are not in the clique and by hypothesis there is a vertex in the $(n+1)$-clique which is connected to all those n vertices. Therefore this vertex is connected to all other $2n$ vertices of the graph. □