Problem 11259
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Proposed by N. Abe (Japan).

For integers n greater that 2, let

$$f(n) = \sum_{j=1}^{n-2} 2^j \sum_S \prod_{k\in S} k,$$

where the sum is over all j-element subsets S of the set $\{1,\ldots,n-1\}$.
Show that $4(2n-1)! + (f(n))^2$ is never the square of an integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$c(n,j) = \sum_S \prod_{k\in S} k$$

where the sum is over all j-element subsets S of the set $\{1,\ldots,n-1\}$. We prove that this number is actually a Stirling number of the first kind

$$c(n,j) = \begin{bmatrix} n \\ n-j \end{bmatrix}.$$

In fact, $c(n,0) = \begin{bmatrix} n \\ n \end{bmatrix} = 1$ and $c(n,n-1) = \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$ for all $n \geq 1$. Moreover for $n,j \geq 2$

$$\begin{aligned} c(n,j) &= \sum_{(n-1)\notin S} \prod_{k\in S} k + \sum_{(n-1)\in S} \prod_{k\in S} k \\ &= c(n-1,j) + (n-1)\cdot c(n-1,j-1) = \begin{bmatrix} n-1 \\ n-1-j \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ n-j \end{bmatrix} = \begin{bmatrix} n \\ n-j \end{bmatrix}. \end{aligned}$$

Hence for $n > 2$ we have that

$$\begin{aligned} f(n) &= \sum_{j=1}^{n-2} 2^j c(n,j) = \sum_{j=1}^{n-2} 2^j \begin{bmatrix} n \\ n-j \end{bmatrix} = 2^n \sum_{j=1}^{n-2} \frac{1}{2^{n-j}} \begin{bmatrix} n \\ n-j \end{bmatrix} \\ &= 2^n \sum_{k=2}^{n-1} \frac{1}{2^k} \begin{bmatrix} n \\ k \end{bmatrix} = 2^n \left(\left(\frac{1}{2}\right)^{\overline{n}} - \frac{1}{2^1}\begin{bmatrix} n \\ 1 \end{bmatrix} - \frac{1}{2^n}\begin{bmatrix} n \\ n \end{bmatrix} \right) \\ &= (2n-1)!! - 2^{n-1}(n-1)! - 1 = (2n-1)!! - (2n-2)!! - 1 \end{aligned}$$

where we used the identity

$$x^{\overline{n}} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Assume that for $n > 2$ there is a number q such that $4(2n-1)! + (f(n))^2 = q^2$.
Letting $d = (2n-1)!!$ and $p = (2n-2)!!$ we have that

$$4(2n-1)! = 4pd = (d+p)^2 - (d-p)^2 = q^2 - (f(n))^2 = q^2 - (d-p-1)^2$$

that is

$$0 < 2d - 2p - 1 = (d-p)^2 - (d-p-1)^2 = (d+p)^2 - q^2 \geq (d+p)^2 - (d+p-1)^2 = 2d + 2p - 1$$

which implies that $4p \leq 0$ and we get a contradiction. □