Problem 11256
(American Mathematical Monthly, Vol.113, November 2006)

Proposed by F. Holland (Ireland).

For complex numbers a, b and c let

$$f(x) = \max\left(\mathrm{Re}(ae^{ix}), \mathrm{Re}(be^{ix}), \mathrm{Re}(ce^{ix})\right).$$

Find $\int_0^{2\pi} f(x)\,dx$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that the answer is the perimeter of the triangle of vertices a, b, c:

$$\int_0^{2\pi} f(x)\,dx = |a-b| + |b-c| + |c-a| = P.$$

Since $a = |a|e^{ix_a}$ for some real number x_a then

$$\int_0^{2\pi} \mathrm{Re}(ae^{ix})\,dx = |a|\int_0^{2\pi} \cos(x-x_a)\,dx = 0 \quad\text{and}\quad \int_0^{2\pi} \left|\mathrm{Re}(ae^{ix})\right|\,dx = |a|\int_0^{2\pi} |\cos(x-x_a)|\,dx = 4|a|.$$

Moreover, $\max(u,v) = (u+v+|u-v|)/2$, therefore letting $f_{a,b}(x) = \int_0^{2\pi} \max\left(\mathrm{Re}(ae^{ix}), \mathrm{Re}(be^{ix})\right)\,dx$,

$$\begin{aligned}\int_0^{2\pi} f_{a,b}(x)\,dx &= \frac{1}{2}\left(\int_0^{2\pi} \mathrm{Re}(ae^{ix})\,dx + \int_0^{2\pi} \mathrm{Re}(be^{ix})\,dx + \int_0^{2\pi} \left|\mathrm{Re}(ae^{ix}) - \mathrm{Re}(be^{ix})\right|\,dx\right)\\ &= 0 + 0 + \frac{1}{2}\int_0^{2\pi} \left|\mathrm{Re}((a-b)e^{ix})\right|\,dx = 2|a-b|.\end{aligned}$$

In a similar way, $\min(u,v) = (u+v-|u-v|)/2$ and letting $g_{a,b}(x) = \int_0^{2\pi} \min\left(\mathrm{Re}(ae^{ix}), \mathrm{Re}(be^{ix})\right)\,dx$,

$$\int_0^{2\pi} g_{a,b}(x)\,dx = -2|a-b|.$$

We note that

$$\max(u,v,w) - \min(u,v,w) = \frac{1}{2}\left[\max(u,v) + \max(v,w) + \max(w,u) - \min(u,v) - \min(v,w) - \min(w,u)\right].$$

Hence, letting $g(x) = \min\left(\mathrm{Re}(ae^{ix}), \mathrm{Re}(be^{ix}), \mathrm{Re}(ce^{ix})\right)$, we have that

$$\int_0^{2\pi} f(x)\,dx - \int_0^{2\pi} g(x)\,dx = \frac{1}{2}\int_0^{2\pi} \left[f_{a,b}(x) + f_{b,c}(x) + f_{c,a}(x)\right]\,dx - \frac{1}{2}\int_0^{2\pi} \left[g_{a,b}(x) + g_{b,c}(x) + g_{c,a}(x)\right]\,dx = 2P.$$

Since $\max(u,v,w) = -\min(-u,-v,-w)$ then

$$\begin{aligned}\int_0^{2\pi} f(x)\,dx &= -\int_0^{2\pi} \min\left(-\mathrm{Re}(ae^{ix}), -\mathrm{Re}(be^{ix}), -\mathrm{Re}(ce^{ix})\right)\,dx\\ &= -\int_0^{2\pi} \min\left(\mathrm{Re}(ae^{i(x+\pi)}), \mathrm{Re}(be^{i(x+\pi)}), \mathrm{Re}(ce^{i(x+\pi)})\right)\,dx = -\int_0^{2\pi} g(x)\,dx\end{aligned}$$

because of the periodicity of the function e^{ix}. Finally we find that

$$\int_0^{2\pi} f(x)\,dx = \frac{1}{2}\left(\int_0^{2\pi} f(x)\,dx - \int_0^{2\pi} g(x)\,dx\right) = P.$$

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