Problem 11253

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Let \( n \) be a positive integer and \( A \) be an \( n \times n \) matrix with all entries \( a_{i,j} \) positive. Let \( P \) be the permanent of \( A \). Prove that

\[
P \geq n! \left( \prod_{1 \leq i,j \leq n} a_{i,j} \right)^{1/n}.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Universit`a di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The permanent \( P \) of an \( n \times n \) matrix \( A \) is defined as follows

\[
P = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}
\]

where \( S_n \) is the set of all the permutations of the integer numbers from 1 to \( n \).

Since \( |S_n| = n! \), by the AM-GM inequality,

\[
P \geq n! \left( \prod_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} \right)^{1/n!} = n! \left( \prod_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} \right)^{1/n!}.
\]

For any \( 1 \leq i \leq n \)

\[
\prod_{\pi \in S_n} a_{i,\pi(i)} = \left( \prod_{j=1}^{n} a_{i,j} \right)^{(n-1)!}
\]

because there are \( (n-1)! \) permutations which have a specific number \( j \) at the \( i \)th position.

Therefore

\[
P \geq n! \left( \prod_{i=1}^{n} \left( \prod_{j=1}^{n} a_{i,j} \right)^{(n-1)!} \right)^{1/n!} = n! \left( \prod_{1 \leq i,j \leq n} a_{i,j} \right)^{1/n}.
\]