Problem 11253
(American Mathematical Monthly, Vol.113, November 2006)

Proposed by D. Beckwith (USA).

Let n be a positive integer and A be an $n \times n$ matrix with all entries $a_{i,j}$ positive. Let P be the permanent of A. Prove that

$$P \geq n! \left(\prod_{1 \leq i,j \leq n} a_{i,j} \right)^{1/n}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The permanent P of an $n \times n$ matrix A is defined as follows

$$P = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}$$

where S_n is the set of all the permutations of the integer numbers from 1 to n.
Since $|S_n| = n!$, by the AM-GM inequality,

$$P \geq n! \left(\prod_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} \right)^{1/n!} = n! \left(\prod_{i=1}^{n} \prod_{\pi \in S_n} a_{i,\pi(i)} \right)^{1/n!}.$$

For any $1 \leq i \leq n$

$$\prod_{\pi \in S_n} a_{i,\pi(i)} = \left(\prod_{j=1}^{n} a_{i,j} \right)^{(n-1)!}$$

because there are $(n-1)!$ permutations which have a specific number j at the ith position.
Therefore

$$P \geq n! \left(\prod_{i=1}^{n} \left(\prod_{j=1}^{n} a_{i,j} \right)^{(n-1)!} \right)^{1/n!} = n! \left(\prod_{1 \leq i,j \leq n} a_{i,j} \right)^{1/n}.$$

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