Problem 11249
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Proposed by D. Beckwith (USA).

A node-labeled rooted tree is a tree such that any parent with label k has $k+1$ children, labeled $1, 2, \dots, k+1$, and such that the root vertex (generation 0) has label k. Find the population of generation n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $a_{n,k}$ be the number of children of generation n with label $k+1$. Since a child of generation n with label k is generated by a parent of generation $n-1$ with label greater or equal to $k-1$ then $a_{0,0} = 1$ and

$$a_{n,k} = \sum_{j=k-1}^{n-1} a_{n-1,j} \quad \text{for } 1 \leq k \leq n$$

that is

$$\begin{bmatrix} a_{1,1} & 0 & 0 & \cdots \\ a_{2,1} & a_{2,2} & 0 & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots \\ a_{1,0} & a_{1,1} & 0 & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The previous recurrence implies that $[a_{n,k}]$ is a *Riordan array* $[f(x), g(x)]$ that is

$$a_{n,k} = [x^n] f(x) \cdot g(x)^k$$

for some formal power series $f(x)$ and $g(x)$ with $g(0) = 0$. The above infinite matrices identity can be therefore written in this way

$$[f(x) \cdot g(x)/x, g(x)] = [f(x), g(x)] \cdot [1/(1-x), x] = [f(x)/(1-g(x)), g(x)].$$

Since for any level $n > 0$ the number of 1s is equal to the number of 2s then $f(x)g(x) = f(x) - 1$ and $g(x) = (f(x) - 1)/f(x)$. Hence

$$f(x) \cdot g(x)/x = (f(x) - 1)/x = f(x)/(1 - g(x)) = f(x)^2$$

that is

$$x f(x)^2 - f(x) + 1 = 0$$

or

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x} = C(x)$$

where $C(x) = \sum_{n=0}^{\infty} C_n x^n$ is the generating function of the Catalan numbers. Finally, the population of generation n, which is the number of 1s at the next level is equal to

$$a_{n+1,0} = [x^{n+1}] f(x) \cdot g(x)^0 = [x^{n+1}] C(x) = C_{n+1} = \frac{1}{n+2} \binom{2(n+1)}{n+1}$$

that is the $(n+1)$th Catalan number. □