Problem 11249
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A node-labeled rooted tree is a tree such that any parent with label \( k \) has \( k + 1 \) children, labeled \( 1, 2, \ldots, k + 1 \), and such that the root vertex (generation 0) has label \( k \). Find the population of generation \( n \).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let \( a_{n,k} \) be the number of children of generation \( n \) with label \( k + 1 \). Since a child of generation \( n \) with label \( k \) is generated by a parent of generation \( n - 1 \) with label greater or equal to \( k - 1 \) then \( a_{0,0} = 1 \) and

\[
a_{n,k} = \sum_{j=k-1}^{n-1} a_{n-1,j} \quad \text{for } 1 \leq k \leq n
\]

that is

\[
\begin{bmatrix}
  a_{1,1} & 0 & 0 & \cdots \\
  a_{2,1} & a_{2,2} & 0 & \cdots \\
  a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
= \begin{bmatrix}
  a_{0,0} & 0 & 0 & \cdots \\
  a_{1,0} & a_{1,1} & 0 & \cdots \\
  a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\cdot
\begin{bmatrix}
  1 & 0 & 0 & \cdots \\
  1 & 1 & 0 & \cdots \\
  1 & 1 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

The previous recurrence implies that \([a_{n,k}]\) is a Riordan array \([f(x), g(x)]\) that is

\[
a_{n,k} = [x^n] f(x) \cdot g(x)^k
\]

for some formal power series \( f(x) \) and \( g(x) \) with \( g(0) = 0 \). The above infinite matrices identity can be therefore written in this way

\[
[f(x) \cdot g(x)/x, g(x)] = [f(x), g(x)] \cdot [1/(1-x), x] = [f(x)/(1-g(x)), g(x)].
\]

Since for any level \( n > 0 \) the number of 1s is equal to the number of 2s then \( f(x)g(x) = f(x) - 1 \) and \( g(x) = (f(x) - 1)/f(x) \). Hence

\[
f(x) \cdot g(x)/x = (f(x) - 1)/x = f(x)/(1 - g(x)) = f(x)^2
\]

that is

\[
xf(x)^2 - f(x) + 1 = 0
\]

or

\[
f(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = C(x)
\]

where \( C(x) = \sum_{n=0}^{\infty} C_n x^n \) is the generating function of the Catalan numbers. Finally, the population of generation \( n \), which is the number of 1s at the next level is equal to

\[
a_{n+1,0} = [x^{n+1}] f(x) \cdot g(x)^0 = [x^{n+1}] C(x) = C_{n+1} = \frac{1}{n+2} \binom{2(n+1)}{n+1}
\]

that is the \((n+1)\)th Catalan number.