Problem 11245
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Proposed by C. Lupu and T. Lupu (Romania).

Consider an acute triangle with sides of lengths $a$, $b$ and $c$ and with an inradius of $r$ and circumradius of $R$. Show that

$$
\frac{r}{R} \leq \frac{\sqrt{2(2a^2 - (b - c)^2)(2b^2 - (c - a)^2)(2c^2 - (a - b)^2)}}{(a + b)(b + c)(c + a)}
$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

We show that the inequality holds for all triangles by using the method described by Stanley Rabinowitz in his paper On The Computer Solutions of Symmetric Homogeneous Triangle Inequalities. We will use the following notation:

$$
[r_1, r_2, r_3] \equiv \sum_{\text{sym}} x^{r_1} y^{r_2} z^{r_3}
$$

where $r_1 \geq r_2 \geq r_3 \geq 0$.

We first transform the symmetric homogeneous triangle inequality into a symmetric homogeneous inequality with variables $x$, $y$, and $z$ by using the identities

$$
R = \frac{abc}{4A}, \quad r = A/s, \quad A^2 = s(s - a)(s - b)(s - c), \quad s = (a + b + c)/2
$$

and via the equations:

$$
a = (y + z)/2, \quad b = (z + x)/2, \quad c = (x + y)/2.
$$

The new variables are positive real numbers because

$$
x = b + c - a, \quad y = c + a - b, \quad z = a + b - c.
$$

Squaring, expanding out and collecting like terms, we obtain the following inequality expressed as simple symmetric sums

$$
$$

This inequality can be easily verified by some applications of Muirhead’s inequality:

$$
9[8, 2, 2] \leq 8[8, 3, 1] + [8, 4, 0],
48[7, 3, 2] \leq 48[7, 4, 1],
88[6, 3, 3] \leq 88[6, 5, 1],
105[4, 4, 4] \leq 66[6, 5, 1] + 7[6, 6, 0] + 24[7, 4, 1] + 8[7, 5, 0].
$$

Note that, again by Muirhead’s inequality, the equality holds if and only if $x = y = z$ that is when the triangle is equilateral. \qed