Problem 11245
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Consider an acute triangle with sides of lengths a, b and c and with an inradius of r and circumradius of R. Show that

$$\frac{r}{R} \leq \frac{\sqrt{2(2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)}}{(a+b)(b+c)(c+a)}$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We show that the inequality holds for *all* triangles by using the method described by Stanley Rabinowitz in his paper *On The Computer Solutions of Symmetric Homogeneous Triangle Inequalities*. We will use the following notation:

$$[r_1, r_2, r_3] \equiv \sum_{\text{sym}} x^{r_1} y^{r_2} z^{r_3}$$

where $r_1 \geq r_2 \geq r_3 \geq 0$.

We first transform the symmetric homogeneous triangle inequality into a symmetric homogeneous inequality with variables x, y, and z by using the identities

$$R = abc/4A\,, \quad r = A/s\,, \quad A^2 = s(s-a)(s-b)(s-c)\,, \quad s = (a+b+c)/2$$

and via the equations:

$$a = (y+z)/2\,, \quad b = (z+x)/2\,, \quad c = (x+y)/2.$$

The new variables are positive real numbers because

$$x = b+c-a\,, \quad y = c+a-b\,, \quad z = a+b-c.$$

Squaring, expanding out and collecting like terms, we obtain the following inequality expressed as simple symmetric sums

$$\begin{aligned}105[4,4,4] + 264[5,4,3] + 88[6,3,3] + 48[7,3,2] + 9[8,2,2] \leq \\ 136[5,5,2] + 106[6,4,2] + 176[6,5,1] + 7[6,6,0] + 72[7,4,1] + 8[7,5,0] + 8[8,3,1] + [8,4,0].\end{aligned}$$

This inequality can be easily verified by some applications of Muirhead's inequality:

$$\begin{aligned}9[8,2,2] &\leq 8[8,3,1] + [8,4,0] \\ 48[7,3,2] &\leq 48[7,4,1] \\ 88[6,3,3] &\leq 88[6,5,1] \\ 264[5,4,3] &\leq 136[5,5,2] + 106[6,4,2] + 22[6,5,1] \\ 105[4,4,4] &\leq 66[6,5,1] + 7[6,6,0] + 24[7,4,1] + 8[7,5,0].\end{aligned}$$

Note that, again by Muirhead's inequality, the equality holds if and only if $x = y = z$ that is when the triangle is equilateral. □