Problem 11241
(American Mathematical Monthly, Vol.113, August-September 2006)

Proposed by R. Tauraso (Italy).

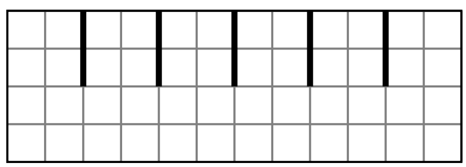
Find a closed formula for

$$\sum_{k=0}^{n} 2^{n-k} \sum_{x\in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}\,,$$

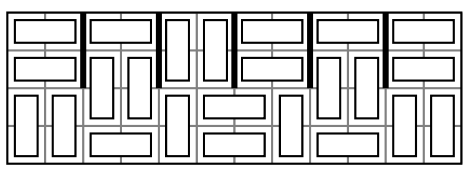
where F_n denotes the nth Fibonacci number (that is, $F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ for $j \geq 2$) and $S[k,n]$ is the set of all $(k+1)$-tuples of nonnegative integers that sum to $n-k$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

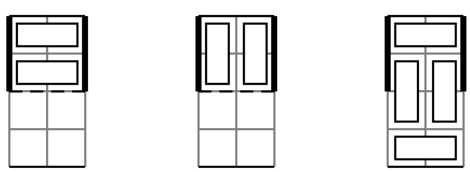
The formula computes the total number a_n of domino tilings of a domain with n *teeth*: each tooth is a 4×2 rectangle and they are glued along the vertical sides. For $n = 6$ the domain is



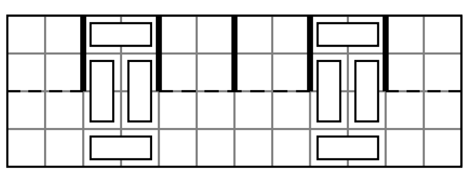
Note that the bold lines can't be crossed by the dominoes: here there is an example of a proper domino tiling



The upper part of a single tooth can be tiled in these three ways



Let k be the number of teeth which are tiled as in the third case. These k teeth induce a partition of the domain: there are the upper parts of the remaining $n-k$ teeth and $k+1$ horizontal strips. For example for $n = 6$ and $k = 2$ the situation could be the following



If we denote by $2x_i$ the size of the gap between the $(i-1)$-th tooth and the i-th tooth then $\sum_{i=1}^{k+1} 2x_i + 2k = 2n$ that is

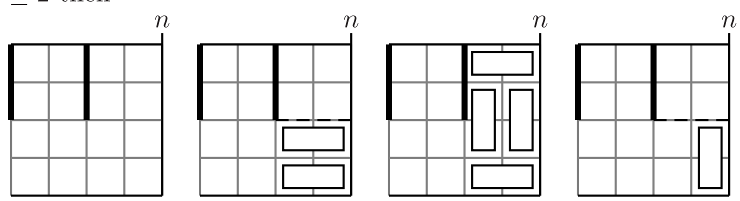
$$\sum_{i=1}^{k+1} x_i = n - k.$$

Moreover any upper part of the $n-k$ teeth can be tiled in 2 ways and the $2 \times 2x_i$ horizontal strip can be tiled in F_{1+2x_i} ways. Therefore the total number of these tilings are

$$a_n = \sum_{k=0}^{n} 2^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}$$

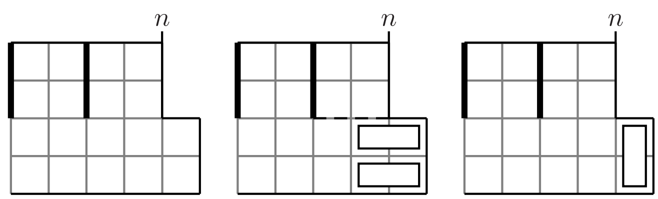
where the internal sum runs over the set $S[k,n]$ of all non negative integer solutions of the equation $x_1 + \cdots + x_{k+1} = n-k$.

Now that we have established this combinatorial interpretation we are going to find a recurrence formula for a_n: if $n \geq 2$ then



$$a_n \quad = \quad 2a_{n-1} \quad + \quad a_{n-1} \quad + \quad 2b_{n-1}$$

and



$$b_n \quad = \quad 2b_{n-1} \quad + \quad a_n$$

that is

$$a_{n+1} = 7a_n - 6a_{n-1}.$$

Since $a(0) = 1$ and $a(1) = 5$ then the required closed formula is

$$a_n = \frac{4 \cdot 6^n + 1}{5}.$$

□