Problem 11240
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Let a, b, and c be the lengths of the sides of a triangle, and let R and r be the circumradius and inradius of the triangle, respectively. Show that

$$\frac{R}{2r} \geq \exp\left(\frac{(a-b)^2}{2c^2} + \frac{(b-c)^2}{2a^2} + \frac{(c-a)^2}{2b^2}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We assume that the triangle is non-degenerate (otherwise $r = 0$). It is well known that

$$R = \frac{abc}{4S}, \quad r = \frac{S}{p}, \quad S = \sqrt{p(p-a)(p-b)(p-c)}$$

where S and p are the area and the semiperimeter of the triangle, respectively. Therefore

$$\frac{R}{2r} = \frac{abcp}{8S^2} = \frac{abc}{(b+c-a)(c+a-b)(a+b-c)}.$$

It suffices to prove that

$$\frac{c}{\sqrt{(b+c-a)(c+a-b)}} \geq \exp\left(\frac{(a-b)^2}{2c^2}\right)$$

then, by symmetry, also these inequalities hold

$$\frac{a}{\sqrt{(c+a-b)(a+b-c)}} \geq \exp\left(\frac{(b-c)^2}{2a^2}\right), \quad \frac{b}{\sqrt{(a+b-c)(b+c-a)}} \geq \exp\left(\frac{(c-a)^2}{2c^2}\right)$$

and taking their product we finally obtain the required inequality. Now,

$$\frac{c}{\sqrt{(b+c-a)(c+a-b)}} = \frac{1}{\sqrt{(1-(a-b)/c)(1+(a-b)/c)}} = \frac{1}{\sqrt{1-(a-b)^2/c^2}}$$

and letting $t = (a-b)^2/c^2 \in [0,1)$ (note that $|a-b| < c$ by the triangle inequality), we have to show that

$$\frac{1}{\sqrt{1-t}} \geq \exp(t/2)$$

that is

$$\log\left(\frac{1}{1-t}\right) \geq t$$

which holds because the graph of the convex function $\log(1/(1-t))$ lies above its tangent line at 0.
□