

Problem 11237

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Prove that the number of 2s occurring in all partitions of n is equal to the number of singletons occurring in all partitions of $n - 1$, where a singleton in a partition is a part occurring once.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If $p(n)$ is the number of integer partitions of n then the generating function of this sequence is

$$P(z) = \sum_{n=0}^{\infty} p(n)z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)} = 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 15z^7 + 22z^8 + o(z^8).$$

In order to compute the number of 2s we introduce a parameter t

$$\frac{1-z^2}{1-tz^2} \cdot P(z)$$

then the generating function which counts the 2s is

$$T(z) = \sum_{n=0}^{\infty} t(n)z^n = \frac{d}{dt} \left(\frac{1-z^2}{1-tz^2} \cdot P(z) \right) \Big|_{t=1} = \frac{z^2}{1-z^2} \cdot P(z) = z^2 + z^3 + 3z^4 + 4z^5 + 8z^6 + 11z^7 + 19z^8 + o(z^8).$$

On the other hand, the generating function which counts the singletons that are equal to $n \geq 1$ is

$$z^n(1-z^n) \cdot P(z)$$

therefore the generating function which counts all the singletons is

$$S(z) = \sum_{n=0}^{\infty} s(n)z^n = \left(\sum_{n=1}^{\infty} z^n(1-z^n) \right) \cdot P(z) = \left(\frac{z}{1-z} - \frac{z^2}{1-z^2} \right) \cdot P(z) = \frac{z}{1-z^2} \cdot P(z).$$

Since $T(z) = zS(z)$ then

$$t(n) = s(n-1).$$