Problem 11237
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Proposed by E. Deutsch (USA).

Prove that the number of 2s occurring in all partitions of n is equal to the number of singletons occurring in all partitions of $n-1$, where a singleton in a partition is a part occurring once.

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If $p(n)$ is the number of integer partitions of n then the generating function of this sequence is

$$P(z)=\sum_{n=0}^{\infty}p(n)z^n=\prod_{n=1}^{\infty}\frac{1}{(1-z^n)}=1+z+2z^2+3z^3+5z^4+7z^5+11z^6+15z^7+22z^8+o(z^8).$$

In order to compute the number of 2s we introduce a parameter t

$$\frac{1-z^2}{1-tz^2}\cdot P(z)$$

then the generating function which counts the 2s is

$$T(z)=\sum_{n=0}^{\infty}t(n)z^n=\frac{d}{dt}\left(\frac{1-z^2}{1-tz^2}\cdot P(z)\right)\Bigg|_{t=1}=\frac{z^2}{1-z^2}\cdot P(z)=z^2+z^3+3z^4+4z^5+8z^6+11z^7+19z^8+o(z^8).$$

On the other hand, the generating function which counts the singletons that are equal to $n\geq 1$ is

$$z^n(1-z^n)\cdot P(z)$$

therefore the generating function which counts all the singletons is

$$S(z)=\sum_{n=0}^{\infty}s(n)z^n=\left(\sum_{n=1}^{\infty}z^n(1-z^n)\right)\cdot P(z)=\left(\frac{z}{1-z}-\frac{z^2}{1-z^2}\right)\cdot P(z)=\frac{z}{1-z^2}\cdot P(z).$$

Since $T(z)=zS(z)$ then

$$t(n)=s(n-1).$$

$\square$