Problem 11234
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Let $a_1, \dots, a_n$ and $b_1, \dots, b_{n-1}$ be real numbers with $a_1 < b_1 < a_2 < \cdots < a_{n-1} < b_{n-1} < a_n$, and let h be an integrable function from $\mathbb{R}$ to $\mathbb{R}$. Show that

$$\int_{-\infty}^{\infty} h\left(\frac{(x-a_1)\cdots(x-a_n)}{(x-b_1)\cdots(x-b_{n-1})}\right)\,dx = \int_{-\infty}^{\infty} h\,(x)\,dx.$$

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It suffices to prove that the Lebesgue measure λ is an invariant measure for the function T

$$T(x) := \frac{(x-a_1)\cdots(x-a_n)}{(x-b_1)\cdots(x-b_{n-1})}$$

that is $\lambda(T^{-1}(A)) = \lambda(A)$ for all measurable set A.
We consider the complex extension of our rational function,

$$T(z) := \frac{(z-a_1)\cdots(z-a_n)}{(z-b_1)\cdots(z-b_{n-1})} = z + \beta + \frac{C_1}{z-b_1} + \cdots + \frac{C_{n-1}}{z-b_{n-1}}$$

where β and $C_1, \dots, C_{n-1}$ are real numbers. Since $a_1 < b_1 < a_2 < \cdots < a_{n-1} < b_{n-1} < a_n$ then every coefficient C_i is negative

$$C_i = \lim_{z\to b_k} (z-b_k)\cdot T(z) = \frac{\overbrace{(b_k-a_1)\cdots(b_k-a_k)}^{\text{positive factors}}\overbrace{(b_k-a_{k+1})\cdots(b_k-a_n)}^{n-k \text{ negative factors}}}{\underbrace{(b_k-b_1)\cdots(b_k-b_{k-1})}_{\text{positive factors}}\underbrace{(b_k-b_{k+1})\cdots(b_k-b_{n-1})}_{n-1-k \text{ negative factors}}} < 0.$$

This property implies that if $\mathrm{Im}(z) > 0$ then

$$\mathrm{Im}(T(z)) = \mathrm{Im}(z) + \sum_{k=1}^{n-1} \mathrm{Im}\left(\frac{C_k}{z-b_k}\right) = \mathrm{Im}(z)\left(1 + \sum_{k=1}^{n-1} \frac{-C_k}{|z-b_k|^2}\right) > 0$$

and therefore also the open upper half-plane $\mathbb{H}$ is an invariant set for T.
The *harmonic measures* on $\mathbb{H}$, which is conformally equivalent to the open unit disc, are the *Cauchy measures* on $\mathbb{R}$ defined by the densities

$$\frac{dP_z}{dx}(x) = \frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{x-z}\right) = \frac{1}{\pi}\,\frac{\mathrm{Im}(z)}{(x-\mathrm{Re}(z))^2 + \mathrm{Im}(z)^2} = \frac{1}{\pi}\,\frac{d}{dx}\arctan\left(\frac{x-\mathrm{Re}(z)}{\mathrm{Im}(z)}\right)$$

and therefore for any $z \in H$ and $x \in \mathbb{R}$

$$P_z \circ T^{-1} = P_{T(z)}.$$

Now, for any measurable set A

$$\lim_{t\to+\infty} \pi t\, P_{it}(T^{-1}(A)) = \lim_{t\to+\infty} t \int_{T^{-1}(A)} \frac{t}{x^2+t^2}\,dx = \lambda(T^{-1}(A)).$$

On the other hand

$$\lim_{t\to+\infty} \pi t\, P_{T(it)}(A) = \lim_{t\to+\infty} t \int_A \frac{\mathrm{Im}(T(it))}{(x-\mathrm{Re}(T(it)))^2 + \mathrm{Im}(T(it))^2}\,dx = \lambda(A)$$

because $\mathrm{Im}(T(it))/t \to 1$ and $|\mathrm{Re}(T(it))|/\mathrm{Im}(T(it)) \to 0$ (∞ is a fixed point of T and $T(it)$ tends to ∞ non-tangentially). Since $P_{it}(T^{-1}(A)) = P_{T(it)}(A)$ then $\lambda(T^{-1}(A)) = \lambda(A)$. $\square$