Problem 11225
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Proposed by J. L. Díaz-Barrero (Spain).

Find

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n \left(\frac{x\log(1+x/n)}{1+x}\right) dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Letting $t = x/n$ the integral becomes

$$\begin{aligned}
\frac{1}{n}\int_0^n \left(\frac{x\log(1+x/n)}{1+x}\right) dx &= \int_0^1 \frac{nt}{1+nt}\cdot\log(1+t)\,dt \\
&= \int_0^1 \log(1+t)\,dt - \int_0^1 \frac{t}{1+nt}\cdot\frac{\log(1+t)}{t}\,dt \\
&= 2\log(2) - 1 - \int_0^1 \frac{t}{1+nt}\cdot\frac{\log(1+t)}{t}\,dt.
\end{aligned}$$

Taking the limit as n goes to infinity then the remaining integral goes to zero because

$$0 \le \int_0^1 \frac{t}{1+nt}\cdot\frac{\log(1+t)}{t}\,dt \le \frac{1}{n}\int_0^1 \frac{\log(1+t)}{t}\,dt < \frac{1}{n}.$$

Actually $\int_0^1 \log(1+t)/t\,dt = \pi^2/12$, but it suffices to note that since $\log(1+t) < t$ for $t > 0$ then this integral is less than 1. Therefore

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n \left(\frac{x\log(1+x/n)}{1+x}\right) dx = 2\log(2) - 1.$$

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