Problem 11222
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Proposed by J. Sondow (USA).

Fix an integer $B \geq 2$, and let $s(n)$ denote the sum of the base-B digits of n. Prove that

$$\prod_{n=0}^{\infty} \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{nB+k}{nB+k+1}\right)^{(-1)^{s(n)}} = \frac{1}{\sqrt{B}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$P := \prod_{n\geq 0} \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{nB+k}{nB+k+1}\right)^{\epsilon(n)} \quad \text{and} \quad Q := \prod_{n\geq 1} \prod_{\substack{k \text{ even} \\ 0\leq k<B}} \left(\frac{nB+k}{nB+k+1}\right)^{\epsilon(n)}$$

where $\epsilon(n) := (-1)^{s(n)}$. It suffices to prove that $P^2 = 1/B$.

Then

$$P\cdot Q = \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{\epsilon(0)} \cdot \prod_{n\geq 1}\left(\frac{nB}{nB+B}\right)^{\epsilon(n)} = \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot \prod_{n\geq 1}\left(\frac{n}{n+1}\right)^{\epsilon(n)}.$$

If $n = mB + k$ with $0 \leq k < B$ then $s(n) = s(B) + k$ and $\epsilon(n) = (-1)^k \cdot \epsilon(m)$

$$\begin{aligned}
\prod_{n\geq 1}\left(\frac{n}{n+1}\right)^{\epsilon(n)} &= \prod_{0<k<B}\left(\frac{k}{k+1}\right)^{\epsilon(k)} \cdot \prod_{m\geq 1}\prod_{0\leq k<B}\left(\frac{mB+k}{mB+k+1}\right)^{\epsilon(mB+k)} \\
&= \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{-1} \cdot \prod_{\substack{k \text{ even} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot \\
&\quad \prod_{m\geq 1}\prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{mB+k}{mB+k+1}\right)^{-\epsilon(m)} \cdot \prod_{m\geq 1}\prod_{\substack{k \text{ even} \\ 0\leq k<B}} \left(\frac{mB+k}{mB+k+1}\right)^{\epsilon(m)} \\
&= \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{-1} \cdot \prod_{\substack{k \text{ even} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot P^{-1}\cdot Q \\
&= \prod_{\substack{k \text{ even} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot P^{-1}\cdot Q
\end{aligned}$$

Therefore

$$P\cdot Q = \prod_{\substack{k \text{ odd} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot \prod_{\substack{k \text{ even} \\ 0<k<B}} \left(\frac{k}{k+1}\right)^{+1} \cdot P^{-1}\cdot Q = \left(\prod_{0<k<B}\frac{k}{k+1}\right)\cdot P^{-1}\cdot Q = \frac{1}{B}\cdot P^{-1}\cdot Q$$

that is $P^2 = 1/B$. Note that all the infinite products are convergent by Abel's theorem. $\square$