Problem 11212
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Proposed by D. Beckwith (USA).

Show that for an arbitrary positive integer n

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=0.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that for $0\leq k\leq n-1$,

$$E(n,k):=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2j}{k}=0.$$

Then the result follows immediately

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=\sum_{r=0}^{n}(-1)^r\binom{n}{n-r}\binom{2(n-r)}{n-1}=(-1)^nE(n,n-1)=0.$$

Note that

$$\begin{aligned}E(n,0) &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}=(1-1)^n=0\quad\text{for } n\geq 1\\ E(n,1) &= \sum_{j=1}^{n}(-1)^j\binom{n}{j}2j=2n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1}=-2n(1-1)^{n-1}=0\quad\text{for } n\geq 2.\end{aligned}$$

Now assume that $E(n-1,k)=0$ for $0\leq k\leq n-2$ and let $2\leq k\leq n-1$ then

$$\begin{aligned}E(n,k) &= -\sum_{j=1}^{n}(-1)^{j-1}\frac{n}{j}\binom{n-1}{j-1}\frac{2j}{k}\binom{2j-1}{k-1}\\ &= -\frac{2n}{k}\sum_{j=1}^{n}(-1)^{j-1}\binom{n-1}{j-1}\binom{2j-1}{k-1}\\ &= -\frac{2n}{k}\sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}\binom{2j+1}{k-1}\\ &= -\frac{2n}{k}\sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}\left[\binom{2j}{k-2}+\binom{2j}{k-1}\right]\\ &= -\frac{2n}{k}\left[E(n-1,k-2)+E(n-1,k-1)\right]=0\end{aligned}$$

and the proof is complete. □