Problem 11208
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Proposed by Li Zhou (USA).

The stage-n Menger sponge M_n is generated recursively, starting with the unit cube M_0. To drill a cube is to partition it into 27 congruent subcubes and remove the closed central cube along with the 6 remaining subcubes sharing a face with that cube, resulting in a solid that is the union of 20 congruent cubes. Given M_{n-1}, construct M_n by drilling each of the 20^{n-1} subcubes of edge-length 3^{1-n} in M_{n-1}. Find the chromatic number of the surface of M_n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first find the number of vertices $v(n)$, sides $s(n)$, and faces $f(n)$ of the tassellation given by the recursive construction of the surface M_n. Since M_0 is the unit cube then $v(0) = 8$, $s(0) = 12$, and $f(0) = 6$. Morever, following the construction, for $n \geq 1$

$$\begin{cases} v(n) = v(n-1) + 2s(n-1) + 4f(n-1) + 8 \cdot 20^{n-1} \\ s(n) = 3s(n-1) + 12f(n-1) + 36 \cdot 20^{n-1} \\ f(n) = 8f(n-1) + 24 \cdot 20^{n-1} \end{cases}$$

In particular $f(n) = 4 \cdot 8^n + 2 \cdot 20^n$ and therefore the *Euler characteristic* is

$$\chi(n) = v(n) - s(n) + f(n) = \chi(n-1) - f(n-1) - 4 \cdot 20^{n-1} = \chi(n-1) - 4 \cdot 8^n - 6 \cdot 20^n.$$

Since the *genus* $g(n) = 1 - \chi(n)/2$ and $g(0) = 0$ then

$$g(n) = g(n-1) + 2 \cdot 8^n + 3 \cdot 20^n = 2 \cdot (8^n - 1)/7 + 3 \cdot (20^n - 1)/19.$$

The first terms of the sequence $g(n)$ are: 0, 5, 81, 1409, 26433.
Finally by Heawood's formula (note that M_n is not a Klein bottle because M_n is orientable) we are able to compute the chromatic number of M_n:

$$\gamma(n) = \left\lfloor \frac{1}{2}\left(7 + \sqrt{48g(n) + 1}\right) \right\rfloor.$$

The first terms of the sequence $\gamma(n)$ are: 4, 11, 34, 133, 566. □

Remark. Note that Heawood's formula was proved by Ringel and Youngs (1968) with two exceptions: the sphere, and the Klein bottle. When the four-color theorem was proved in 1976 then the formula was confirmed also for the sphere. As regards the only exception left, the Klein bottle, Heawood's formula gives 7 but the correct is 6 as demonstrated by the Franklin graph. See for example *Four Colors Suffice : How the Map Problem Was Solved* by Robin Wilson.