Problem 11206
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Find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}^2$$

where $\{x\}=x-\lfloor x\rfloor$ *denotes the fractional part of* x.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}^2=\sum_{k=1}^{n}\left\{\frac{1}{k/n}\right\}^2\cdot\frac{1}{n}$$

is a Riemann sum corresponding to the integral

$$\int_0^1\left\{\frac{1}{x}\right\}^2 dx.$$

Since $n\leq 1/x<n+1$ implies that

$$\left\{\frac{1}{x}\right\}=\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor=\frac{1}{x}-n$$

then

$$\int_0^1\left\{\frac{1}{x}\right\}^2 dx=\lim_{N\to\infty}\sum_{n=1}^{N}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\left(\frac{1}{x}-n\right)^2 dx$$

Now since $H_n=\log(n)+\gamma+o(1)$ and $n!=n^n e^{-n}\sqrt{2\pi n}(1+o(1))$ then

$$\begin{aligned}
\sum_{n=1}^{N}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\left(\frac{1}{x}-n\right)^2 dx &= \sum_{n=1}^{N}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\left(\frac{1}{x^2}-\frac{2n}{x}+n^2\right)dx\\
&= \sum_{n=1}^{N}\left[-\frac{1}{x}-2n\log(x)+n^2x\right]_{\frac{1}{n+1}}^{\frac{1}{n}}\\
&= \sum_{n=1}^{N}\left(1+2n\log(n)-2n\log(n+1)+1-\frac{1}{n+1}\right)\\
&= \sum_{n=1}^{N}\left(2+2\left(n\log(n)-(n+1)\log(n+1)\right)+2\log(n+1)-\frac{1}{n+1}\right)\\
&= 2N-2(N+1)\log(N+1)+2\log((N+1)!)+1-H_{N+1}\\
&= 2N+2\log((N+1)!/(N+1)^{N+1})+1-\log(N+1)-\gamma+o(1)\\
&= 2N+2\log(e^{-(N+1)}\sqrt{2\pi(N+1)}(1+o(1)))+1-\log(N+1)-\gamma+o(1)\\
&= 2N-2(N+1)+\log(2\pi)+\log(N+1)+1-\log(N+1)-\gamma+o(1)\\
&= \log(2\pi)-\gamma-1+o(1).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}^2=\int_0^1\left\{\frac{1}{x}\right\}^2 dx=\lim_{N\to\infty}\sum_{n=1}^{N}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\left(\frac{1}{x}-n\right)^2 dx=\log(2\pi)-\gamma-1\approx 0.2606614021.$$

□