**Problem 11187**
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*Find a closed formula for the number of ways to tile a* 4 *by* $n$ *rectangle with* 1 *by* 2 *dominoes.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

There is a well known formula for the number of domino tilings of a $m$ by $n$ rectangle due to P. W. Kasteleyn (see *The statistics of dimers on a lattice*, Physica, 27 (1961), 1209–1225)

$$T(m,n)=\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{j\pi}{m+1}\right)+4\cos^2\left(\frac{k\pi}{n+1}\right)\right)^{1/4}$$

Since

$$\left(\frac{\sin((n+1)x)}{\sin(x)}\right)^2=\prod_{k=1}^{n}\left(4x^2-4\cos^2\left(\frac{k\pi}{n+1}\right)\right)$$

then

$$T(m,n)=\prod_{j=1}^{m}\left|\frac{\sin\left((n+1)\arccos\left(i\cos\left(\frac{j\pi}{m+1}\right)\right)\right)}{\sin\left(\arccos\left(i\cos\left(\frac{j\pi}{m+1}\right)\right)\right)}\right|^{1/2}.$$

If $m=4$ then

$$\cos(\pi/5)=-\cos^2(4\pi/5)=(1+\sqrt{5})/4 \quad\text{and}\quad \cos(2\pi/5)=-\cos^2(3\pi/5)=(1-\sqrt{5})/4$$

and

$$T(4,n)=\prod_{j=1}^{4}\left|\frac{\sin\left((n+1)\arccos\left(i\cos\left(j\pi/5\right)\right)\right)}{\sin\left(\arccos\left(i\cos\left(j\pi/5\right)\right)\right)}\right|^{1/2}=\frac{\prod_{j=1}^{2}\left|\sin\left((n+1)\arccos\left(i\cos\left(j\pi/5\right)\right)\right)\right|}{\prod_{j=1}^{2}\left|\sin\left(\arccos\left(i\cos\left(j\pi/5\right)\right)\right)\right|}.$$

Note that $\sin(\arccos(ix))=\sqrt{1+x^2}$ and therefore

$$\prod_{j=1}^{2}\left|\sin\left(\arccos\left(i\cos\left(j\pi/5\right)\right)\right)\right|=\sqrt{1+\cos^2(\pi/5)}\sqrt{1+\cos^2(2\pi/5)}=\sqrt{29}/4.$$

Moreover if $n$ is even

$$|\sin((n+1)\arccos(ix))|=|\cosh((n+1)\operatorname{arcsinh}(x))|=\frac{1}{2}\left((x+\sqrt{1+x^2})^{n+1}+(x+\sqrt{1+x^2})^{-(n+1)}\right)$$

on the other hand if $n$ is odd

$$|\sin((n+1)\arccos(ix))|=|\sinh((n+1)\operatorname{arcsinh}(x))|=\frac{1}{2}\left((x+\sqrt{1+x^2})^{n+1}-(x+\sqrt{1+x^2})^{-(n+1)}\right).$$

hence we get the final formula:

$$T(4,n)=\left(a^{n+1}-(-1/a)^{n+1}\right)\cdot\left(b^{n+1}-(-1/b)^{n+1}\right)/\sqrt{29}$$

where

$$a=\left(1+\sqrt{5}+\sqrt{22+2\sqrt{5}}\right)/4>1 \quad\text{and}\quad b=4/\left(1-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right)>1.$$

Note that $ab$, $1/ab$, $-b/a$ and $-a/b$ are the four zeros of the polynomial $x^4-x^3-5x^2-x+1$ related to the recurrence:

$$T(4,n)=T(4,n-1)+5T(4,n-2)+T(4,n-3)-T(4,n-4).$$

The first terms of the sequence $T(n,4)$ are (see A005178 in *The On-Line Encyclopedia of Integer Sequences*):

$$1,5,11,36,95,281,781,2245,6336,18061,51205,145601,413351,1174500,3335651,9475901$$

and

$$\lim_{n\to\infty} T(4,n)/T(4,n-1)=ab=(1+\sqrt{29}+\sqrt{14+2\sqrt{29}}))/4\approx 2.84053619409.$$

□