Problem 11183
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Proposed by D. Beckwith (USA).

The left and right pillars of a triumphal arch are each built of blocks of height 1 or 2. Blocks of height 2 may not sit upon blocks of height 1. How many designs are feasible if the lintel must sit upon the pillars and if exactly n blocks must be used in the construction of the pillars?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let l_1, l_2, r_1 and r_2 be respectively the number of 1-blocks and the number of 2-blocks in the left pillar and in the right pillar. The problem is therefore equivalent to find the number of non-negative integer solutions of

$$\begin{cases} l_1 + l_2 + r_1 + r_2 = n \\ l_1 + 2l_2 = r_1 + 2r_2 \end{cases}$$

We first count the solutions such that $l_2 \geq r_2$, that is $d := l_2 - r_2 \geq 0$. Then $r_1 = l_1 + 2(l_2 - r_2) = l_1 + 2d$ is a non-negative integer if l_1 and d are non-negative integers and therefore it suffices to count the non-negative integer solutions of

$$n = l_1 + l_2 + r_1 + r_2 = l_1 + (d + r_2) + (l_1 + 2d) + r_2 = 2l_1 + 2r_2 + 3d$$

that is the coefficient of x^n of the generating function

$$[x^n]\left(\left(\sum_{k=1}^{n} x^{2k}\right)^2 \cdot \left(\sum_{k=1}^{n} x^{3k}\right)\right) = [x^n]\left(\frac{1}{(1-x^2)^2 \cdot (1-x^3)}\right).$$

In particular if $l_2 = r_2$ then also $l_1 = r_1$ and $n = 2l_1 + 2r_1$ and in this case the number of solutions is

$$[x^n]\left(\left(\sum_{k=1}^{n} x^{2k}\right)^2\right) = [x^n]\left(\frac{1}{(1-x^2)^2}\right).$$

Finally, by the inclusion-exclusion principle the total number of solutions is given by

$$[x^n]\left(2 \cdot \frac{1}{(1-x^2)^2 \cdot (1-x^3)} - \frac{1}{(1-x^2)^2}\right) = [x^n]\left(\frac{1+x^3}{(1-x^2)^2 \cdot (1-x^3)}\right)$$

which means that the sequence (Sloane's sequence A008806) can be computed by the recurrence

$$a_n = 2a_{n-2} + a_{n-3} - a_{n-4} - 2a_{n-5} + a_{n-7} \quad \text{for } n \geq 7$$

with $a_0 = 1$, $a_1 = 0$, $a_2 = a_3 = 2$, $a_4 = 3$, $a_5 = 4$, $a_6 = 6$.
An explicit formula can be easily found solving the above linear recurrence

$$a_n = (n^2 + 4n + 12 - 9 \cdot [n \equiv 1 \mod 2] - 8 \cdot [n \equiv 1 \mod 3])/12.$$

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