Problem 11182

Proposed by S. Amrahov (Turkey).

Let \( \langle a_n \rangle \) be an arithmetic progression of positive integers for which the common difference is prime. Given that the sequence includes both a term that is a perfect \( j \)th power and a term that is a perfect \( k \)th power, and that \( j \) and \( k \) are relatively prime, prove that there exists a term that is a perfect \( jk \)th power.

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The term of the arithmetic progression is given by
\[
a_n = a_0 + n \cdot p
\]
for some prime \( p \) and for \( n \in \mathbb{N} \).

By hypothesis there are \( n_1, n_2 \in \mathbb{N} \) such that
\[
a_0 + n_1 \cdot p = x^j \quad \text{and} \quad a_0 + n_2 \cdot p = y^k
\]
for some positive integers \( x \) and \( y \).

If \( a_0 \) is a multiple of \( p \), say \( a_0 = c \cdot p \) then
\[
x^{jk} = (a_0 + n_1 \cdot p)^k = ((c + n_1)^k \cdot p^{k-1}) \cdot p = a_0 + ((c + n_1)^k \cdot p^{k-1} - c) \cdot p
\]
and since \( ((c + n_1)^k \cdot p^{k-1} - c) \) is a non negative integer the perfect \( jk \)th power \( x^{jk} \) belongs to the arithmetic progression.

Now assume that \( a_0 \) is not a multiple of \( p \). Since \( a_0 = x^j = y^k \mod p \) then for \( s, t \in \mathbb{Z} \)
\[
a_0^{ks+jt} = a_0^s \cdot a_0^t = x^{ks} \cdot y^{kt} = (x^s \cdot y^t)^{jk} \mod p.
\]

Since \( j \) and \( k \) are relatively prime there are some \( s_0, t_0 \in \mathbb{Z} \) such that \( ks_0 + jt_0 = 1 \). Moreover, by Euler-Fermat theorem \( a^{p-1} = 1 \mod p \) (\( a_0 \) is not a multiple of \( p \)), hence letting \( s = s_0 + a \cdot (p - 1) \)
and \( t = t_0 + b \cdot (p - 1) \) for \( a, b \in \mathbb{N} \) we have that
\[
(x^s \cdot y^t)^{jk} = a_0^{ks+jt} = a_0^{1+(a+b)(p-1)} = a_0 \mod p.
\]

Therefore there is an integer \( n_3 \) such that
\[
a_0 + n_3 \cdot p = (x^s \cdot y^t)^{jk}.
\]

and by taking the integers \( a \) and \( b \) big enough we can be sure that \( s \) and \( t \) are non negative. Hence the perfect \( jk \)th power \((x^s \cdot y^t)^{jk}\) belongs to the arithmetic progression because it is greater than \( a_0 \)
and therefore \( n_3 \) is non negative. \( \square \)