Problem 11182

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Let \(< a_n \rangle \) be an arithmetic progression of positive integers for which the common difference is prime. Given that the sequence includes both a term that is a perfect \( j \)th power and a term that is a perfect \( k \)th power, and that \( j \) and \( k \) are relatively prime, prove that there exists a term that is a perfect \( jk \)th power.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The term of the arithmetic progression is given by \( a_n = a_0 + n \cdot p \) for some prime \( p \) and for \( n \in \mathbb{N} \).

By hypothesis there are \( n_1, n_2 \in \mathbb{N} \) such that
\[
(a_0 + n_1 \cdot p) = x^j, \quad a_0 + n_2 \cdot p = y^k
\]
for some positive integers \( x \) and \( y \).

If \( a_0 \) is a multiple of \( p \), say \( a_0 = c \cdot p \) then
\[
x^{jk} = (a_0 + n_1 \cdot p)^k = ((c + n_1)^k \cdot p^{k-1}) \cdot p = a_0 \cdot ((c + n_1)^k \cdot p^{k-1} - c) \cdot p
\]
and since \((c + n_1)^k \cdot p^{k-1} - c\) is a non negative integer the perfect \( jk \)th power \( x^{jk} \) belongs to the arithmetic progression.

Now assume that \( a_0 \) is not a multiple of \( p \). Since \( a_0 = x^j = y^k \mod p \) then for \( s, t \in \mathbb{Z} \)
\[
a_0^{ks+jt} = a_0^s \cdot a_0^t = x^{k \cdot s} \cdot y^{k \cdot t} = (x^s \cdot y^t)^{jk} \mod p.
\]

Since \( j \) and \( k \) are relatively prime there are some \( s_0, t_0 \in \mathbb{Z} \) such that \( ks_0 + jt_0 = 1 \). Moreover, by Euler-Fermat theorem \( a^{p-1} = 1 \mod p \) (\( a_0 \) is not a multiple of \( p \)), hence letting \( s = s_0 + a \cdot (p-1) \) and \( t = t_0 + b \cdot (p-1) \) for \( a, b \in \mathbb{N} \) we have that
\[
(x^s \cdot y^t)^{jk} = a_0^{ks+jt} = a_0^{1+(a+b)(p-1)} = a_0 \mod p.
\]

Therefore there is an integer \( n_3 \) such that
\[
a_0 + n_3 \cdot p = (x^s \cdot y^t)^{jk}.
\]

and by taking the integers \( a \) and \( b \) big enough we can be sure that \( s \) and \( t \) are non negative. Hence the perfect \( jk \)th power \((x^s \cdot y^t)^{jk}\) belongs to the arithmetic progression because it is greater than \( a_0 \) and therefore \( n_3 \) is non negative.