Problem 11182
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Proposed by S. Amrahov (Turkey).

Let $\langle a_n \rangle$ be an arithmetic progression of positive integers for which the common difference is prime. Given that the sequence includes both a term that is a perfect jth power and a term that is a perfect kth power, and that j and k are relatively prime, prove that there exists a term that is a perfect jkth power.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The term of the arithmetic progression is given by $a_n = a_0 + n \cdot p$ for some prime p and for $n \in \mathbb{N}$. By hypothesis there are $n_1, n_2 \in \mathbb{N}$ such that

$$a_0 + n_1 \cdot p = x^j \quad \text{and} \quad a_0 + n_2 \cdot p = y^k$$

for some positive integers x and y.
If a_0 is a multiple of p, say $a_0 = c \cdot p$ then

$$x^{jk} = (a_0 + n_1 \cdot p)^k = \left((c + n_1)^k \cdot p^{k-1}\right) \cdot p = a_0 + \left((c + n_1)^k \cdot p^{k-1} - c\right) \cdot p$$

and since $\left((c + n_1)^k \cdot p^{k-1} - c\right)$ is a non negative integer the perfect jkth power x^{jk} belongs to the arithmetic progression.
Now assume that a_0 is not a multiple of p. Since $a_0 = x^j = y^k \mod p$ then for $s, t \in \mathbb{Z}$

$$a_0^{ks+jt} = a_0^{ks} \cdot a_0^{jt} = x^{jks} \cdot y^{kjt} = \left(x^s \cdot y^t\right)^{jk} \mod p.$$

Since j and k are relatively prime there are some $s_0, t_0 \in \mathbb{Z}$ such that $ks_0 + jt_0 = 1$. Moreover, by Euler-Fermat theorem $a^{p-1} = 1 \mod p$ (a_0 is not a multiple of p), hence letting $s = s_0 + a \cdot (p-1)$ and $t = t_0 + b \cdot (p-1)$ for $a, b \in \mathbb{N}$ we have that

$$\left(x^s \cdot y^t\right)^{jk} = a_0^{ks+jt} = a_0^{1+(a+b)(p-1)} = a_0 \mod p.$$

Therefore there is an integer n_3 such that

$$a_0 + n_3 \cdot p = \left(x^s \cdot y^t\right)^{jk}.$$

and by taking the integers a and b big enough we can be sure that s and t are non negative. Hence the perfect jkth power$\left(x^s \cdot y^t\right)^{kj}$ belongs to the arithmetic progression because it is greater than a_0 and therefore n_3 is non negative. $\square$