Problem 11179
(American Mathematical Monthly, Vol.112, October 2005)

Proposed by D. Beckwith (USA).

For positive integers i and j let

$$m_{i,j} = \begin{cases} -1 & \text{if } j \mid (i+1) \\ 0 & \text{if } j \nmid (i+1) \end{cases},$$

and when $n \geq 2$ let M_n be the $(n-1) \times (n-1)$ matrix with (i,j)-entry $m_{i,j}$. Evaluate $\det(M_n)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We show that for $n \geq 2$ the determinant of M_n is equal to the value of the Möbius function at n:

$$\det(M_n) = \mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct prime factors} \end{cases}.$$

We will use the Iverson bracket notation

$$[\text{proposition}] = \begin{cases} 1 & \text{if the proposition is true} \\ 0 & \text{if the proposition is false} \end{cases}.$$

Let $n = \prod_{k=1}^r p_k^{\alpha_k}$ be the prime factorization of $n \geq 2$ then

$$[j = n] = \prod_{k=1}^r ([j|n] - [j|n/p_k])$$

because if j is a proper divisor of n then at least for one prime p_k, $p_k^{\alpha_k}$ does not divide j and therefore the factor $[j|n] - [j|n/p_k] = 1 - 1 = 0$.
On the other hand, by expanding the product, since $[j|a] \cdot [j|b] = [j|\gcd(a,b)]$, then

$$[j = n] = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} \cdot [j|n/\prod_{k \in I} p_k]$$

where the sum is taken over all subsets of the set $\{1, \ldots, r\}$.
Now we modify the last row of M_n by adding a proper linear combination of the other rows. This operation does not change the determinant of M_n, but it will help us to compute it.
If n is not the product of r distinct prime factors then we set for $j = 1, \ldots, n-1$

$$\begin{aligned} m'_{(n-1),j} &:= \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} \cdot m_{(n/\prod_{k \in I} p_k - 1),j} \\ &= -\sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} \cdot [j|n/\prod_{k \in I} p_k] = -[j = n] = 0. \end{aligned}$$

Therefore the new last row is identically zero and $\det(M_n) = 0 = \mu(n)$.
If n is the product of r distinct prime factors, that is $n = \prod_{k=1}^r p_k$, then we set for $j = 1, \ldots, n-1$

$$\begin{aligned} m'_{(n-1),j} &:= \sum_{I \subsetneq \{1,\ldots,r\}} (-1)^{|I|} \cdot m_{(n/\prod_{k \in I} p_k - 1),j} \\ &= -\sum_{I \subsetneq \{1,\ldots,r\}} (-1)^{|I|} \cdot [j|n/\prod_{k \in I} p_k] = -[j = n] + (-1)^r \cdot [j|n/\prod_{k=1}^r p_k] \\ &= -[j = n] + (-1)^r \cdot [j|1] = (-1)^r \cdot [j = 1]. \end{aligned}$$

Hence the entries of the new last row are all zero but one, that is $m'_{(n-1),1} = (-1)^r$.
In order to find the determinant we expand the computation along the last row:

$$\det(M_n) = (-1)^{(n-1)+1} \cdot m'_{(n-1),1} \cdot \det(T_n)$$

where T_n is the $(n-2) \times (n-2)$ matrix obtained by eliminating the last row and the first column from the matrix M_n. The main diagonal entries of T_n are $m_{i,i+1} = -[i|i] = -1$ for $i = 1, \dots, n-2$. The entries above the main diagonal of T_n are $m_{i,j} = -[j|i+1] = 0$ because $j > i+1$. Hence T_n is a lower-triangular matrix and $\det(T_n) = (-1)^{n-2}$. Finally

$$\det(M_n) = (-1)^n \cdot (-1)^r \cdot (-1)^{n-2} = (-1)^r = \mu(n).$$

□