Problem 11176
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Proposed by O. Faynshteyn (Germany).

Let α, β, and γ be the angle measures of a non-degenerate triangle. Show that

$$\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha + \cos\beta + \cos\gamma} \geq -2$$

with equality if and only if the triangle is equilateral.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we prove that when n is an odd integer then

$$\cos(n\alpha) + \cos(n\beta) + \cos(n\gamma) = 1 + 4(-1)^{(n-1)/2} \sin(n\alpha/2)\sin(n\beta/2)\sin(n\gamma/2).$$

Since $\gamma = \pi - \alpha - \beta$ then

$$\begin{aligned}
\cos(n\alpha) + \cos(n\beta) + \cos(n\gamma) &= 2\cos(n(\alpha+\beta)/2)\cos(n(\alpha-\beta)/2) + \left(1 - 2\sin^2(n\gamma/2)\right)\\
&= 2\cos(n(\alpha+\beta)/2)\cos(n(\alpha-\beta)/2) + \left(1 - 2\cos^2(n(\alpha+\beta)/2)\right)\\
&= 1 + 2\cos(n(\alpha+\beta)/2)\cdot(\cos(n(\alpha-\beta)/2) - \cos(n(\alpha+\beta)/2))\\
&= 1 + 2\sin(n\pi/2)\sin(n\gamma/2)\cdot 2\sin(n\alpha/2)\sin(n\beta/2)\\
&= 1 + 4(-1)^{(n-1)/2}\sin(n\alpha/2)\sin(n\beta/2)\sin(n\gamma/2).
\end{aligned}$$

Assume that $n = 1$. Let a, b and c be the side lenghts and let s be the semi-perimeter. It is well known that the area A of the triangle can be computed in the following ways

$$A = \sqrt{s(s-a)(s-b)(s-c)} = sr = \frac{abc}{4R}$$

where r and R are respectively the inradius and the circumradius of the triangle. Then

$$\begin{aligned}
\cos(\alpha) + \cos(\beta) + \cos(\gamma) &= 1 + 4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)\\
&= 1 + 4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\\
&= 1 + 4\frac{(s-a)(s-b)(s-c)}{abc} = 1 + \frac{r}{R}.
\end{aligned}$$

By the famous Euler's inequality $R \geq 2r$ with equality if and only if the triangle is equilateral. Hence it follows that

$$1 < \cos(\alpha) + \cos(\beta) + \cos(\gamma) = 1 + \frac{r}{R} \leq \frac{3}{2}.$$

Now we go back to our inquality. Since $\cos(\alpha) + \cos(\beta) + \cos(\gamma)$ is positive, it is equivalent to

$$1 - 4\sin(3\alpha/2)\sin(3\beta/2)\sin(3\gamma/2) \geq -2\left(1 + 4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)\right).$$

Noting that $\sin(3x/2) = \sin(x/2)(1 + 2\cos(x))$, it suffices to prove that

$$[4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)]\cdot[(1+2\cos(\alpha))(1+2\cos(\beta))(1+2\cos(\gamma)) - 2] \leq 3$$

with equality if and only if the triangle is equilateral. It is easy to estimate the first group of factors

$$0 < 4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2) = \cos(\alpha) + \cos(\beta) + \cos(\gamma) - 1 \leq \frac{3}{2} - 1 = \frac{1}{2}$$

with equality if and only if the triangle is equilateral.
Moreover, if $(1+2\cos(\alpha))<0$ then $2\pi/3<\alpha<\pi$ and therefore since α is obtuse then the other two factors $(1+2\cos(\beta))$ and $(1+2\cos(\gamma))$ are positive and

$$(1+2\cos(\alpha))(1+2\cos(\beta))(1+2\cos(\gamma))<0.$$

So assume that all the factors $(1+2\cos(\alpha))$, $(1+2\cos(\beta))$ and $(1+2\cos(\gamma))$ are nonnegative. Then by the AGM inequality

$$(1+2\cos(\alpha))(1+2\cos(\beta))(1+2\cos(\gamma)) \leq \left(1+\frac{2}{3}(\cos(\alpha)+\cos(\beta)+\cos(\gamma))\right)^3 \leq \left(1+\frac{2}{3}\cdot\frac{3}{2}\right)^3 = 8$$

with equality if and only if the triangle is equilateral.
Finally we have proved our claim because

$$[4\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2)]\cdot[(1+2\cos(\alpha))(1+2\cos(\beta))(1+2\cos(\gamma))-2] \leq \frac{1}{2}\cdot(8-2)=3$$

with equality if and only if the triangle is equilateral. $\square$