**Problem 11175**
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Proposed by W. Calbeck (USA).

*Let $m$ and $n$ be integers. Show that there are exactly four (distinct) integer solutions to $|x^2-mx| = n$ if and only if there exist integers $p$, $q$, and $s$ such that $n = s^2pq(p+q)(p+2q)$, $m = s((p+q)(p+2q)-pq)$ and $p$, $q$, $p+q$, $p+2q$ and $s$ are different from zero.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If $n = s^2pq(p+q)(p+2q)$ and $m = s((p+q)(p+2q)-pq)$ then

$$x^2-mx-n = (x+spq)\cdot(x-s(p+q)(p+2q)) \quad \text{and} \quad x^2-mx+n = (x-sp(p+q))\cdot(x-sp(p+2q))$$

and therefore the following integer numbers

$$x_1 = s(p+q)(p+2q), \quad x_2 = -spq, \quad x_3 = sp(p+q), \quad x_4 = sq(p+2q)$$

are the four solutions. Note that they are distinct iff $p$, $q$, $p+q$, $p+2q$ and $s$ are different from zero. Now assume that the solutions of the equation $|x^2-mx| = n$, that is

$$x_1 = (m+\sqrt{m^2+4n})/2,\ x_2 = (m-\sqrt{m^2+4n})/2,\ x_3 = (m+\sqrt{m^2-4n})/2,\ x_4 = (m+\sqrt{m^2-4n})/2$$

are integers then there exist integers $z$ and $w$ such that

$$m^2+4n = z^2, \quad m^2-4n = w^2 \quad \text{and} \quad m = z = w \mod 2.$$

If we multiply the first two equations we find

$$(zw)^2 + (4n)^2 = (m^2)^2$$

that is $(zw, 4n, m^2)$ is a Pythagorean triple and therefore there are integers $u$ and $v$ such that

$$m^2 = u^2 + v^2.$$

This means that also $(u, v, m)$ is a Pythagorean triple and there are integers $p$, $q$ and $s$ such that

$$m = s((p+q)^2+q^2) = s((p+q)(p+2q)-pq), \quad u = 2s(p+q)q \quad \text{and} \quad v = s((p+q)^2-q^2) = sp(p+2q).$$

Going back to the Pythagorean triple $(zw, 4n, m^2)$ we have two cases: either

$$4n = u^2-v^2 = (u+v)(u-v) = s^2(2q^2-p^2)(p^2+4pq+2q^2)$$

which is impossible because otherwise $z^2 = m^2+4n = 8s^2q^2(p+q)^2$ and $w^2 = m^2-4n = 2s^2p^2(p+2q)^2$ ($\sqrt{2}$ is an irrational number) or

$$n = 2uv/4 = s^2pq(p+q)(p+2q).$$

□